INTENSE MAGNETIC FIELDS IN ASTROPHYSICS

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Abstract. In this paper we summarize the current knowledge of research on the influence of intense magnetic fields on physical processes. The contents are summarized in the enclosed Table of Contents.

Table of Contents:

1. Introduction

The presence of magnetic fields in nature is a common phenomenon. Our earth possesses an approximate dipole field aligned at about 15° from its rotational axis with a strength of $\frac{1}{2}$ G, and, according to fossil evidence, the field has an ancient history. The Sun, which is an average star, possesses a magnetic field of apparently complicated structure and configuration. The average value of the solar field (on the surface of the Sun) is 1 G, but this average is derived from a very heterogeneous distribution of fields ranging from zero to several thousands of gauss in sunspots. Many stars are also known to possess magnetic fields with strengths in excess of 500 G, which is the present lower limit of detectability of stellar magnetic fields (C67)[†]. In one case the average field strength is in excess of 3.4×10^4 G, about twice the saturation field of iron! Our Galaxy also possesses a magnetic field whose strength is a few times 10^{-6} G.

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The energy density of the field is comparable to the kinetic energy density of gas in our Galaxy, and the galactic field is believed to have a nonnegligible effect on the structure of our Galaxy (W64a, S66).

The role played by magnetic fields in astrophysics has been extensively discussed (S66, C67, W64, Ca67, CC68d, CC68e). Most fields discussed have strengths much less than 10^5 G and such fields will hereafter be referred to as classical fields since for such fields the quantum effect is not likely to play any significant role in astrophysics. Exceptions are: (1) low temperature physics with a cryogenic temperature; (2) Zeeman splitting of atomic lines, which has been discussed elsewhere. Although on occasion fields more intense than 10^5 G can still be regarded as classical, in no case can quantum effects be entirely neglected when the field strength is greater than 10^8 G.

In the following sections we will briefly summarize the properties of classical magnetic fields, in particular, the source of magnetic fields in astronomical objects and the flux conservation law. In the remaining part of this paper, we will be concerned with the quantum effects of magnetic fields in astrophysics.

2. The Source of Magnetic Fields in Astronomical Objects

As far as we know, magnetic fields can only be generated and maintained by one of the following processes:

(1) Moving charges (electric current).

(2) Alignment of spin magnetic moment.

(3) Alignment of magnetic moment due to orbital angular momentum of some types of atoms.

(4) Landau Orbital Magnetized State.

The presence of an electric current in a conductor can generate a magnetic field according to the Maxwell equation:

$$\operatorname{curl} H = \frac{4\pi}{c} j \,. \tag{2.1}$$

However, the current density is subject to Ohmic dissipation. In conductors of ordinary size (e.g., coils in a small transformer) the Ohmic dissipation will dissipate a current completely in a matter of milliseconds. In the case of astronomical objects such as stars and nebulae, the conductivity is so high and the inductance so large that the time of decay ranges from millions to billions of years.

On the other hand, in order to align the spin or orbital magnetic moment, invariably a solid crystalline structure is needed. Further, the temperature cannot exceed the Curie temperature, which is of the order of 10^3 K. As a result, although (2) and (3) are important processes in solid matter possessing permanent or semipermanent magnetism, they are of negligible importance in astronomical objects. The new process (4) which can give rise to a semipermanent magnetic field in dense bodies such as neutron stars and white dwarfs, will be discussed in a separate section.

A current is made of moving charges. In a system in thermal equilibrium in which the distribution of velocities is random and isotropic there is no net velocity in any direction, and consequently, a system in complete thermodynamic equilibrium does not possess a magnetic field. A magnetic field will be present if one of the charges (usually electrons) in a medium possesses a small net drifting velocity v_d. Let us take a simple configuration, that of a cylinder of charged particles rotating with an angular velocity ω which in turn gives rise to an average linear drift velocity v_d (in cm/sec). Ley ϱ be the density of matter in the cylinder (in g/cm^3) and Z be the average number of free charge per atom of average mass number A. Then the field at the test point is given by Equation (2.1). With a simple calculation one can show that the expression for the field H is given by (R in cm)

$$H/H_q \cong 10^2 \frac{Z}{A}\left(\frac{v}{c}\right)\varrho R\,, \quad H_q = \frac{m^2c^3}{e\hbar} = 4.414 \times 10^{13}\ \mathrm{G}\,. \tag{2.2}$$

The Planck constant $\hbar$ has been purposely introduced only for comparison with quantum regime. Here R is the linear dimension over which the drifting charges extend. If $\varrho = 1$, $Z/A = 1$, $R \simeq 0.1$, then a field of 10^4 G can be generated by a relatively small drifting velocity of only 1 cm/sec. This is to be compared with the thermal velocity of the electrons v_{th} which is approximately ($T_4 \equiv 10^{-4}\,T$)

$$v_{\text{th}}/c \simeq 10^{-3}T_4^{1/2} \tag{2.3}$$

where T is the temperature. As another example, let us put in (2.2) the conditions appropriate to a neutron star:

$$\varrho = 10^{14}\ \mathrm{g/cm^3}\,, \qquad Z/A \simeq 10^{-3}\,, \qquad v \simeq 1\ \mathrm{cm/sec} \tag{2.4}$$

we find

$$H \cong 10^{15}\ \mathrm{G}\,.$$

Thus, the existence of a relatively strong field implies only a negligible departure from a state of complete thermodynamic equilibrium.

The origin of magnetic fields has been a knotty problem in astrophysics. It is generally believed that turbulence is the cause for macroscopic drifting velocity for one of the two component charges in a plasma, but up-to-date theory of turbulence is still very crude and does not lend any insight to the solutions of the problem (LL60). The condition for turbulence to exist requires a high Reynolds number which is usually satisfied on low density plasma such as interstellar plasma. It is thus believed that magnetic fields observed in stars originated from their prestellar state.

Once a current is established the flux is determined by the subsequent events of development and by the dissipation. The dissipation is usually small, the time for decay being of the order of millions and billions of years. If the subsequent development of the astronomical object carrying a magnetic field follows a simple scaling law, then the magnetic field is roughly proportional to the square of the scaling factor.

This is easily shown as follows. The Maxwell equation is

$$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c} \mathbf{j} \tag{2.5}$$

where $\mathbf{j}$, the current density is equal to the density of charge ϱ_e times the drifting velocity. Under a transformation $r \to \alpha r$ where α is a scaling factor, then the density of charge ϱ_e increases as α^{-3}. Equations (2.5) is invariant if $\mathbf{H} \to \alpha^{-2}\,\mathbf{H}$. Thus the invariance of the Maxwell equation requires that $|H| \propto r^{-2}$ (this result can also be derived differently fron flux conservation law of a closed loop of current).

3. Classical and Quantizing Fields

In a magnetic field a charged particle suffers a force perpendicular to the field and the direction of motion. The equation of motion is:

$$\dot{\mathbf{p}} = e\mathbf{E} + \frac{e}{c} \mathbf{v} \times \mathbf{H} \tag{3.1}$$

where $\mathbf{E}$ is the electric field, $\mathbf{p}$ is the momentum, and $\mathbf{v} \times \mathbf{H}$ is the magnetic force. Consider now the case $E=0$. In this case the force is always perpendicular to the instantaneous direction of the motion, and hence no work is done on the particle. As a consequence the energy of a particle in a magnetic field is invariant.

In a constant uniform magnetic field along the z-axis the solution of (3.1) is readily obtained as (LL62)

$$\begin{aligned} x &= x_0 + R_L \sin(\omega t + \alpha) \\ y &= y_0 + R_L \cos(\omega t + \alpha) \\ z &= z_0 + v_z t \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &R_L = v_\perp/\omega_L = \frac{v_\perp E}{ecH} = \frac{cp_\perp}{eH} = \gamma v mc/eH = \left(\frac{H}{H_q}\right)^{-1} \gamma\,(v/c)\,(\hbar/mc) \\ &R_L = \left(\frac{H}{H_q}\right)^{-1} \gamma\beta\lambda\!\!\!^{-}_c\,, \qquad \omega_L = ecH/E\,, \qquad \gamma^{-2} = 1 - (v/c)^2 \\ &H_q = m^2c^3/e\hbar\,, \qquad\qquad \omega_L = (H/H_q)\,(E/mc^2)^{-1}\,(mc^2/\hbar) \end{aligned} \tag{3.3}$$

Here $\lambda\!\!\!^{-}_c$ is the Compton wavelength for the electron (3.8615×10^{-11} cm). The trajectories are helices with a constant velocity v_z in the z-direction. $v_\perp$ is the velocity of the particle in the plane perpendicular to the field and E is the total energy of the electron:

$$E = \gamma mc^2 \qquad v^2 = v_\perp^2 + v_z^2 = v_x^2 + v_y^2 + v_z^2 \tag{3.4}$$

R_L is the radius of the orbit projected in the x, y-plane and ω_L is the angular frequency of the circular motion in the x, y-plane (the Larmor frequency). (x_0, y_0) is the center of the orbit (the *guiding center*). The presence of a magnetic field therefore confines the motion of the charges particle in the x, y-plane.

If the field is not uniform and if the non-uniformity is small (i.e. $1/H|\text{grad}\,H|^{-1} \gg R_L$) then it has been shown that if grad H is in the x, y-plane, the guiding center slowly moves toward regions of weaker fields. If $|\text{grad}\,H|$ is along the field line, then as the particle moves toward regions of weaker fields the energy in the xy-plane is slowly transferred to that in the parallel direction and as the particle moves toward regions of stronger fields the energy in the parallel direction is transferred into that in the perpendicular plane. Reflections of particles can take place if the magnetic field gradient is sufficiently large.

We shall assume that the gradient of the magnetic field is small and that the circular motion of the electron is unaffected by the non-uniformity of the field.* As the field strength increases R_L decreases and ω_L increases. Consider now the de-Broglie wavelength of a particle

$$\lambda_B = \frac{h}{p} = \frac{h}{m\gamma v} = \frac{2\pi\lambda_c}{\gamma\beta}.$$

Comparing it with the Larmor radius R_L, Equation (3.3) we found that quantum effects are important where

$$R_L \leqslant \lambda_B$$

or

$$\gamma^2\beta^2 \equiv \frac{\beta^2}{1-\beta^2} \lesssim 2\pi\frac{H}{H_q}$$

we have

$$\frac{v}{c} = \left[\frac{2\pi k\,(H/H_q)}{1 + 2\pi k\,(H/H_q)}\right]^{1/2} \qquad k \geqslant 1\,.$$

We see that a particle with a velocity of 10^7 cm/sec ($\simeq 1$ eV) enters in the quantum domain at $H \simeq 10^7$ G. The classical equation of motion (3.1), from which the classical trajectories of a free electron are derived, is therefore expected to be invalid. The solution for the trajectory then calls for the use of the Schrodinger equation or the Dirac equation. According to these solutions the circular orbits in a magnetic field are quantized.

The quantized total electron energy is (R28, P30, H31, JL49, CC68a, b, c)

$$E\,(p_z, n) \equiv E = mc^2\left[1 + (p_z/mc)^2 + 2nH/H_q\right]^{1/2}. \tag{3.5}$$

$n = 0, 1, 2, \ldots \infty$ is the principal quantum number which characterizes the sizes of the orbits, which are referred to as *Landau levels*. Comparing Equation (3.5) with the

* In the case of quantizing fields the Larmor orbit is $\sim$ de-Broglie wavelength $\leqslant 10^{-8}$ cm and the Larmor period is $\leqslant 10^{-8}$ sec. If the field gradient is less than $H/(\hbar/mc)$ and the time rate of change is less than $H/(\hbar/mc^2)$, all fields can be regarded as time constant and homogeneous in space.

usual expression for the energy of an electron

$$E = mc^2 \left[1 + (p_z/mc)^2 + (p_x/mc)^2 + (p_y/mc)^2\right]^{1/2} \tag{3.6}$$

we find that quantization replaces the x- and y-momenta $p_x^2 + p_y^2$ by the quantity $2n(H/H_q)\, m^2c^2$. In other words the momentum of the electron in the plane perpendicular to the field is quantized into a discrete set according to the formula:

$$p_\perp^2 \equiv p_x^2 + p_y^2 = 2n\,(H/H_q)\, m^2c^2\,. \tag{3.7}$$

This discrete set becomes continuous as $n \to \infty$. According to Bohr's correspondence principle, when n is large, classical trajectories of the electrons can be applied. However, for small values of n the discrete behavior of the perpendicular ($\perp$) momentum must be taken into account. If we assume that the parallel ($\|$) momentum is approximately the same as the $\perp$-momentum, then the criteria for application of the classical trajectory is when

$$(p_z/mc)^2 \gg 2\,(H/H_q)\,. \tag{3.8}$$

If we can assign a temperature to the electron, then in the nonrelativistic case $p_z^2/2m \sim kT$, and Equation (3.8) becomes:

$$kT/mc^2 \gg H/H_q \tag{3.9}$$

or, numerically,

$$T\,(\mathrm{K}) \gg 10^{-4}\, H\,(\mathrm{G})\,. \tag{3.10}$$

That is, at $H = 10^{11}$ G, classical trajectory analysis is inapplicable even at a temperature of 10^7 K.

In the relativistic limit $p_z c \sim kT$ and Equation (3.8) becomes:

$$(kT/mc^2)^2 \gg 2\,(H/H_q)\,. \tag{3.11}$$

At a field of 10^{13} G even at a temperature of 5.9×10^9 K quantization of orbits have to be taken into account.

In a degenerate medium p_z is replaced by the Fermi momentum p_F. A similar analysis may be made on the applicability of classical trajectory.

A. STRONG COUPLING REGIME

An electron in a magnetic field can make transitions from one orbit to another accompanied by the emission of a photon. Such radiation is called the synchrotron radiation (to be discussed later). In classical regimes the energy loss per orbit is usually negligible, and in computing the trajectories (3.2), the energy loss can be treated as a small perturbation.

In a strong field or when the electron energy is high the energy loss rate will be so large that a substantial amount of energy of the electron will be lost before a complete orbit is described. This is the regime of the strong coupling. Two effects must then be taken into account.

(a) The perturbation theory of computing energy loss rate is not applicable and the complete wave function must be used to evaluate the energy loss rate and classical trajectories cannot be used. This has been worked out by Klepikov (K54) and his result has been extensively discussed by Erber (E66).

(b) The radiation reaction must be taken into account. This has been discussed to some extent by Erber, but no quantitative result is available yet.

This regime is characterized by the condition that the lifetime of electrons against synchrotron radiation loss is smaller or comparable to the Larmor period. From the expression of the Larmor period and the lifetime against synchrotron radiation (to be discussed later), we find that the condition for strong coupling is ($\gamma = E/mc^2$)

$$\Gamma \equiv \frac{e^2}{\hbar c} \gamma^2 \frac{H}{H_q} \gtrsim 1 \quad \text{or} \quad \gamma^2 H \geqslant 6 \times 10^{15} .$$

4. The Impossibility of Spontaneous Pair Creation in a Magnetic Field

It has often been erroneously stated that in a field greater than $2H_q = 8.8 \times 10^{13}$ G, spontaneous pair creation can take place. The basis of this argument is as follows. In a magnetic field the non-relativistic spin interaction hamiltonian is given by $\sim (q\hbar/2mc)\,\boldsymbol{\sigma}\cdot\mathbf{H}$ where $q = -e$ for electron and $q = e$ for positron, with $e > 0$. The non-relativistic magnetic energy of an electron with its spin antiparallel to the field is $-\frac{1}{2}e\hbar/mc\,H$. The corresponding quantity for a positron parallel to the field is $\frac{1}{2}e\hbar/mc\,H$. When H is greater than $2H_q = 2m^2c^3/e\hbar$ the non-relativistic magnetic energy is then greater than $2mc^2$, and it was thus concluded that a pair of electron and positrons would be created with spins opposite to each other properly aligned with the field. However, according to Equation (3.5) the lowest state of energy of an electron in a magnetic field remains unchanged and the separation between the positive and negative energy states is still $2mc^2$. Therefore, spontaneous pair creation cannot take place. When the anomalous magnetic moment of the electron is taken into account (TBZ66, CCFC68), Equation (3.5) is changed to ($s = \pm 1$)

$$E = mc^2 \left\{ x^2 + \left[\sqrt{1 + 2nH/H_q} + s \frac{\alpha}{4\pi} H/H_q \right]^2 \right\}^{1/2}, \qquad x = p_z/mc .$$

When $x = 0$, $n = 0$ for $\alpha H/4\pi H_q = 1$ or $H \cong 10^{16}$ G, the lowest energy eigenvalue is zero and therefore spontaneous pair creation could occur. This conclusion is however invalidated by the fact that for high magnetic field, the form of the anomalous magnetic moment cannot be taken to be simply $\alpha/2\pi$. This point has been emphasized by Jancovici (J69). An asymptotic expression for the anomalous magnetic moment when $H \gg H_q$ can be found in (TBBD69).

5. Thermodynamic Properties

As is well known in classical plasma physics the presence of a magnetic field introduces an anisotropy only from the magnetic stress (LL62)

$$T_{\alpha\beta} = \frac{1}{4\pi}\left[- E_\alpha E_\beta - H_\alpha H_\beta + \tfrac{1}{2}\delta_{\alpha\beta}\left(E^2 + H^2\right)\right] \tag{5.1}$$

leaving the equation of state unchanged.* This is because the radius of the classical orbits takes continuous values and all values of $\perp$-momentum are available. In the case of strong quantization, the $\perp$-momentum must assume the quantized values given in Equation (3.7).

The equation of state for a gas of electrons may be obtained by evaluating the energy momentum tensor (AB65)

$$T_{\mu\nu} = \tfrac{1}{2}\hbar c\left[\bar{\psi}\gamma_\mu \partial_\nu \psi - \partial_\nu \bar{\psi}\gamma_\mu \psi\right] \tag{5.2}$$

using the exact wave function solutions of the Dirac (R28, P30, H31, CC68, L149, R52, S60) or Schrodinger (K27, D28, L30, P30, UY30) equation. This has been done in a previous paper (CC68a). In this paper we will present an alternative approach based on a re-interpretation of the velocity and the generalization of the usual procedure in deriving the equation of state (CCFC68a).

Because of the cylindrical symmetry of the problem, $T_{xx} = T_{yy}$. If $P_\perp$ is the pressure in the direction perpendicular to the field and $P_\parallel$ the pressure in the parallel direction, then

$$P_\perp = \langle T_{xx}\rangle = \langle T_{yy}\rangle \qquad P_\parallel = \langle T_{zz}\rangle \tag{5.3}$$

where the symbol $\langle \ldots \rangle$ stands for a sum of the contribution of all particles according to the distribution function $f(p)$.

The pressure is the force exerted on the wall of a container during the reflection of particles of velocity v from the wall. In a reflection the exchange of momentum is $2p_x$ and the rate of collision is $v_x/2 = \frac{1}{2}\partial E/\partial p_x = \frac{1}{2}c^2 P_x/E$, hence (in the x-direction, for example)

$$P_\perp = \langle 2p_x \tfrac{1}{2} v_x\rangle = \left\langle \frac{c^2 p_x^2}{E}\right\rangle = \left\langle \frac{c^2 p_y^2}{E}\right\rangle \tag{5.4}$$

$$P_\parallel = \left\langle \frac{c^2 p_z^2}{E}\right\rangle. \tag{5.5}$$

In the quantized case we have

$$c^2 p_x^2 + c^2 p_y^2 \to m^2 c^2 (H/H_q)\, 2n \tag{5.6}$$

* As is well known, in the viscosity-free case all diagonal elements vanish and T_{xx}, T_{yy}, T_{zz} are pressure in the x, y, and z direction.

hence we can write Equations (5.4) and (5.5) as

$$P_{\perp} = \left\langle \frac{c^2 p_x^2}{E} \right\rangle = \left\langle \frac{c^2 p_y^2}{E} \right\rangle = \tfrac{1}{2} m^2 c^4 \frac{H}{H_q} \left\langle \frac{2n}{E} \right\rangle \tag{5.7}$$

$$P_{\parallel} = \left\langle \frac{c^2 p_z^2}{E} \right\rangle. \tag{5.8}$$

The statistical average, as indicated by the symbol $\langle \ldots \rangle$ is achieved by multiplying the quantity of interest by the Fermi distribution function $f(p_z, n)$

$$f\,(p_z, n) = \{1 + \exp \tilde{\beta} \left[E\,(p_z, n) - \tilde{\mu} \right]\}^{-1} \tag{5.9}$$

where $\tilde{\beta} = (kT)^{-1}$, T is the temperature and $\tilde{\mu}$ is the chemical potential plus the electron rest mass, mc^2.

The summation over p_z in the average $\langle \ldots \rangle$ is carried out as usual, that is,

$$\sum_{p_z} \to \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} \mathrm{d}p_z \tag{5.10}$$

and the summation over p_x and p_y is carried out as follows:

$$(2\pi\hbar)^2 \sum_{p_x} \sum_{p_y} \to \int_{-\infty}^{+\infty} \mathrm{d}p_x \int_{-\infty}^{+\infty} \mathrm{d}p_y = \int_0^{\infty} p_{\perp}\, \mathrm{d}p_{\perp} \int_0^{2\pi} \mathrm{d}\phi = \pi \int_0^{\infty} \mathrm{d}p_{\perp}^2 \tag{5.11}$$

where $\phi = \tan^{-1} p_y/p_x$. Quantization requires that

$$p_{\perp}^2 \to m^2 c^4 (H/H_q)\, 2n \tag{5.12}$$

hence

$$\int_0^{\infty} \mathrm{d}p_{\perp}^2 \to \sum_{n=0}^{\infty} \omega_n \tag{5.13}$$

where ω_n is the degeneracy of the level n. ω_n is evaluated as follows: when $H = 0$, the number of levels in $\mathrm{d}p_x$ and $\mathrm{d}p_y$ at p_x and p_y is given by

$$(2\pi\hbar)^{-2}\, \mathrm{d}p_x\, \mathrm{d}p_y\,. \tag{5.14}$$

In the presence of H these levels coalesce into those of a harmonic oscillator, as shown in Figure 1. The degeneracy of each of these levels is therefore given by integrating Equation (5.14) as follows [K65]

$$\omega_n = (2\pi\hbar)^{-2} \int_{A < p_{\perp}{}^2 < B} \mathrm{d}p_x\, \mathrm{d}p_y$$

where [see Equation (3.7)]

$$A = m^2c^2 \frac{H}{H_q} 2n \qquad B = m^2c^2 \frac{H}{H_q} 2(n+1).$$

$$\frac{\mathrm{d}p_x\,\mathrm{d}p_y}{(2\pi\hbar)^2}\,.$$

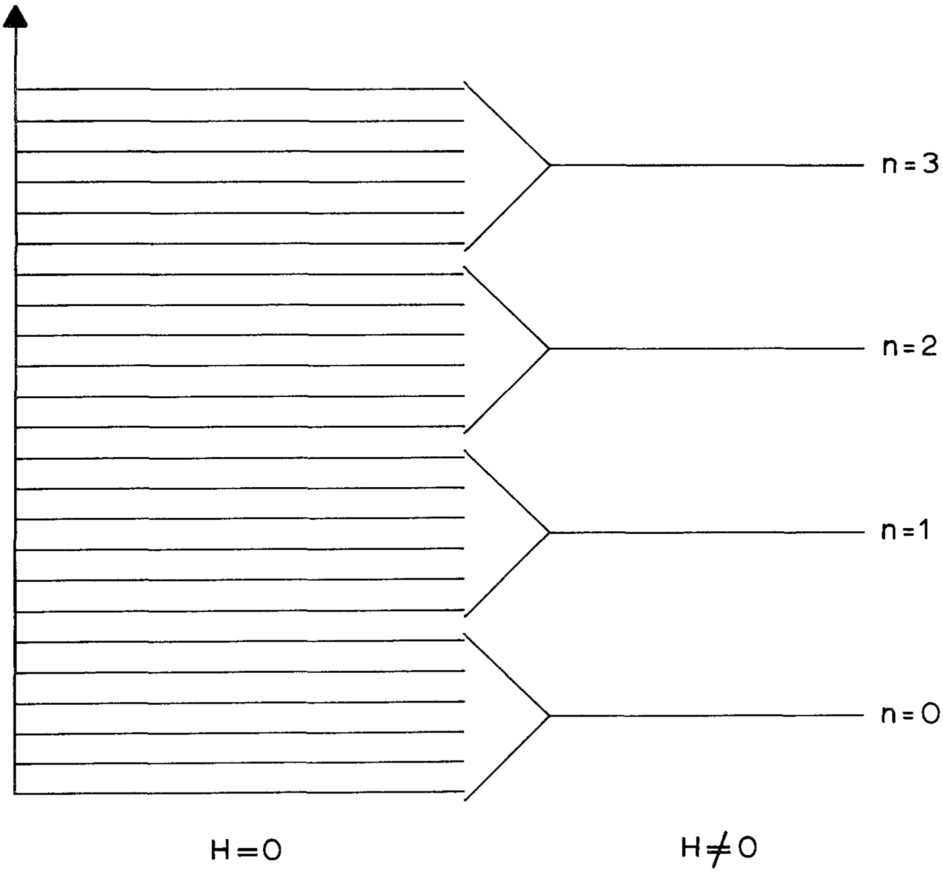

Fig. 1. The coalescence of free particle states into equally spaced harmonic oscillator energy states in the presence of a magnetic field.

Introducing the cylindrical coordinates $(p_\perp, \phi)$, we then obtain:

$$\begin{aligned} \omega_n &= (2\pi\hbar)^{-2} \int_0^{2\pi} \mathrm{d}\phi \int_{A<p_\perp^2<B} p_\perp\,\mathrm{d}p_\perp = 2\pi(2\pi\hbar)^{-2}\,\tfrac{1}{2}(B-A) \\ \omega_n &= \frac{1}{2\pi}(\hbar/mc)^{-2}\,(H/H_q)\,. \end{aligned} \tag{5.15}$$

Equations (5.7) and (5.8) therefore become:

$$P_{\perp} = \frac{1}{8\pi^2} \frac{mc^2}{\lambda_c^3} \left(\frac{H}{H_q}\right)^2 \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} \mathrm{d}\,(p_z/mc) \frac{2n\, mc^2}{E} f\,(p_z, n) \tag{5.16}$$

$$P_{\parallel} = \frac{1}{8\pi^2} \frac{mc^2}{\lambda_c^3} \left(\frac{H}{H_q}\right) \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} \mathrm{d}\,(p_z/mc) \frac{c^2 p_z^2}{E} f\,(p_z, n)\,. \tag{5.17}$$

On the first look it might be deduced that $P_{\perp} \propto H^2$ and $P_{\parallel} \propto H$. However, since H also appears inside the integrals the functional dependence of P on H is rather complicated and it will be discussed later. The energy density U and the particle density N are obtained in the usual manner as follows:

$$U = \sum_{p_x} \sum_{p_y} \sum_{p_z} (E - mc^2)\, f\,(p_z, n) \tag{5.18}$$

$$U = \frac{1}{4\pi^2} \frac{mc^2}{\lambda_c^3} \frac{H}{H_q} \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} \frac{E - mc^2}{mc^2} f\,(p_z, n)\, \mathrm{d}\,(p_z/mc) \tag{5.19}$$

$$N = \frac{1}{4\pi^2} \frac{1}{\lambda_c^3} \frac{H}{H_q} \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} f\,(p_z, n)\, \mathrm{d}\,(p_z/mc)\,. \tag{5.20}$$

These thermodynamic functions appear to be complicated, but they can be simplified considerably by noting that there is a degeneracy between the levels $n, s = +1$ and $n+1, s = -1$. This amounts to saying that all the levels are doubly degenerate except the one with $n = 0$.

$$P_{xx} = P_{yy} = P_0 \theta^2 \sum_{n=1}^{\infty} n \int_0^{\infty} \frac{f\,(x, n)\, \mathrm{d}x}{\varepsilon\,(x, n, \theta)} \tag{5.21}$$

$$P_{zz} = P_0 \theta \left[\tfrac{1}{2} \int_0^{\infty} \frac{x^2 f\,(x, 0)\, \mathrm{d}x}{\varepsilon\,(x, 0, \theta)} + \sum_{n=1}^{\infty} \int_0^{\infty} \frac{f\,(x, n)\, x^2\, \mathrm{d}x}{\varepsilon\,(x, n, \theta)} \right] \tag{5.22}$$

$$U = U_0 \theta \left[\tfrac{1}{2} \int_0^{\infty} f\,(x, 0)\, \varepsilon\,(x, 0, \theta)\, \mathrm{d}x + \sum_{n=1}^{\infty} \int_0^{\infty} f\,(x, n)\, \varepsilon\,(x, n, \theta)\, \mathrm{d}x \right] \tag{5.23}$$

$$N = N_0 \theta \left[\tfrac{1}{2} \int_0^{\infty} f\,(x, 0)\, \mathrm{d}x + \sum_{n=1}^{\infty} \int_0^{\infty} f\,(x, n)\, \mathrm{d}x \right] \tag{5.24}$$

where

$$f^{-1}(x, n) = 1 + \exp\{[\varepsilon(x, n, \theta) - \mu]/(T/T_0)\}$$
$$\varepsilon(x, n, \theta) = 1 + x^2 + 2n\theta, \qquad \theta = H/H_q, \qquad \mu = \tilde{\mu}/mc^2$$
$$P_0 = U_0 = \frac{1}{\pi^2}\frac{mc^2}{\lambda_c^3} = 1.4407 \times 10^{24}\ \text{erg/cm}^3 = 1.4407 \times 10^{24}\ \text{dyn/cm}^2$$
$$N_0 = \frac{1}{\pi^2}\frac{1}{\lambda_c^3} = 1.7598 \times 10^{30}\ \text{cm}^{-3}, \qquad T_0 = mc^2/k = 5.903 \times 10^9\,\text{K}. \tag{5.25}$$

By means of the following transformations (CC68b)

$$v = x/a_n \qquad a_n^2 = 1 + 2n\theta$$
$$\varepsilon(x, n, \theta) = a_n(1 + v^2)^{1/2} \tag{5.26}$$

the equations of state can be expressed in terms of the following functions:

$$C_1(T, \mu) = \int_0^\infty \frac{f(\mu, T, v)}{\sqrt{1 + v^2}}\,\mathrm{d}v \tag{5.27}$$

$$C_2(T, \mu) = \int_0^\infty \frac{v^2}{\sqrt{1 + v^2}} f(\mu, T, v)\,\mathrm{d}v \tag{5.28}$$

$$C_3(T, \mu) = \int_0^\infty (1 + v^2)^{1/2} f(\mu, T, v)\,\mathrm{d}v = C_1(T, \mu) + C_2(T, \mu) \tag{5.29}$$

$$C_4(T, \mu) = \int_0^\infty f(\mu, T, v)\,\mathrm{d}v \tag{5.30}$$

where

$$f(\mu, T, v) = \left[1 + \exp\left(\frac{\sqrt{1 + v^2} - \mu}{T/T_0}\right)\right]^{-1}. \tag{5.31}$$

The results are (CC68b)

$$P_{xx} = P_{yy} = P_0\theta^2 \sum_{n=1}^{\infty} n\, C_1(T/a_n, \mu/a_n) \tag{5.32}$$

$$P_{zz} = P_0\theta\left[\tfrac{1}{2} C_2(T, \mu) + \sum_{n=1}^{\infty} a_n^2\, C_2(T/a_n, \mu/a_n)\right] \tag{5.33}$$

$$U = U_0\theta\left[\tfrac{1}{2} C_3(T, \mu) + \sum_{n=1}^{\infty} a_n^2\, C_3(T/a_n, \mu/a_n)\right] \tag{5.34}$$

$$N = N_0\theta\left[\tfrac{1}{2} C_4(T, \mu) + \sum_{n=1}^{\infty} a_n\, C_4(T/a_n, \mu/a_n)\right]. \tag{5.35}$$

The C_k functions are average values of dynamic variables of a one-dimensional gas. $C_1(T, \mu)$ is the average value of E^{-1} where $E^{-1}=(1+v^2)^{1/2}$ is the total energy of a one-dimensional particle of unit mass and momentum v. $C_2(T, \mu)$ is the average value of $v\,\mathrm{d}v/\mathrm{d}E$ whose statistical average gives the pressure of a one-dimensional gas. C_3 is the average of E and C_4 is the average particle density (in appropriate units). The properties of a magnetized Fermi gas therefore are closely related to those of a one-dimensional gas. This is intuitively clear since an electron in a magnetic field is quantized in energy in the $\perp$-direction and moves freely only in the $\|$-direction.

The properties of the C_k functions have been studied extensively in the general case of a Fermi gas [CC68b]. No simple inversion formula expressing T and μ in terms of C_k's is known. In the following we will study 2 cases, the non-degenerate and the degenerate case.

In the nondegenerate case the factor 1 can be neglected in the denominator of the integrand of the C_k's. Since the relativistic case is always marked by some degree of degeneracy (because of pair creation at relativistic temperatures) we will consider the non-relativistic case, $v \ll 1$. In this case ($\mu' = \mu - 1$)

$$f(\mu, T, v) = \exp[\mu'/(T/T_0)] \exp[-v^2/(2T/T_0)] \tag{5.36}$$

and

$$\begin{aligned} C_1(T, \mu) &= \int_0^\infty (1+v^2)^{1/2} \exp[\mu'/(T/T_0)] \exp[-v^2/(2T/T_0)]\,\mathrm{d}v \\ &\cong \sqrt{\pi T/2T_0}\, \exp[\mu'/(T/T_0)] \end{aligned} \tag{5.37}$$

$$\begin{aligned} C_2(T, \mu) &= \int_0^\infty v^2 \exp[\mu'/(T/T_0)] \exp[-v^2/(2T/T_0)]\,\mathrm{d}v \\ &= (\pi T/2T_0)^{1/2} (T/T_0) \exp[\mu'/(T/T_0)] \end{aligned} \tag{5.38}$$

$$\begin{aligned} C_3(T, \mu) &\simeq \int_0^\infty (1 + \tfrac{1}{2}v^2) \exp[\mu'/(T/T_0)] \exp[-v^2/(2T/T_0)]\,\mathrm{d}v \\ &= (\pi T/2T_0)^{1/2} (1 + T/2T_0) \exp[\mu'/(T/T_0)] \end{aligned} \tag{5.39}$$

$$C_4(T, \mu) = C_1(T, \mu) = (\pi T/2T_0)^{1/2} \exp[\mu'/(T/T_0)]\,. \tag{5.40}$$

We therefore find:

$$P_{xx} = P_{yy} = P_0 \theta^2 \sum_{n=1}^{\infty} n (\pi T/2T_0 a_n)^{1/2} \exp[(\mu T_0/T) - (T_0 a_n/T)]\,. \tag{5.41}$$

In the non-relativistic case $T \ll T_0$ and the requirement of non-degeneracy implies $\mu T_0 \simeq T$. Let us consider the case $n\theta \ll 1$ so that

$$a_n = (1 + 2n\theta)^{1/2} \simeq 1 + n\theta \simeq 1\,.$$

Then

$$P_{xx} = P_{yy} = P_0\theta^2 (\pi T/2T_0)^{1/2} \exp[\mu'/(T/T_0)] \sum_{n=1}^{\infty} n \exp(-n\theta T_0/T)$$
$$= P_0\theta^2 (\pi T/2T_0)^{1/2} \exp[\mu'/(T/T_0)] \exp(\theta T_0/T) [\exp(\theta T_0/T) - 1]^{-2}. \tag{5.42}$$

Similarly

$$P_{zz} = P_0\theta (\pi/2)^{1/2} (T/T_0)^{3/2} \exp(\mu T_0/T) \left\{\frac{1}{2} + \frac{\exp(\theta T_0/T)}{\exp(\theta T_0/T) - 1}\right\}. \tag{5.43}$$

The internal energy density is a little bit more involved. Note that $a_n^2 = 1 + 2n\theta$ and $a_n^2 \cong 1$, hence

$$U/U_0 = \theta \left\{\tfrac{1}{2}[C_3(T, \mu) - C_4(T, \mu)] + \sum_{n=1}^{\infty} [a_n^2 C_3(T/a_n, \mu/a_n) - a_n C_4(T/a_n, \mu/a_n)]\right\}$$
$$\cong \theta\sqrt{\pi} (T/2T_0)^{3/2} \exp(\mu' T_0/T) \times \left\{\tfrac{1}{2} + [\exp(\theta T_0/T) - 1]^{-1} + 2\sqrt{\pi}\frac{T_0\theta}{T}\frac{\exp(\theta T_0/T)}{[\exp(\theta T_0/T) - 1]^2}\right\}. \tag{5.44}$$

Analogously

$$N = N_0\theta\sqrt{\pi} (T/2T_0) \exp[\mu'/(T/T_0)] [\tfrac{1}{2} + \{\exp(\theta T_0/T) - 1\}^{-1}]. \tag{5.45}$$

From quantum statistics the ratio of particles in states separated by an energy ΔE is $\exp - (\Delta E/kT)$. The energy separation between adjacent Landau levels is θmc^2 and $\Delta E/kT$ becomes $\theta T_0/T$. When $\theta T_0/T \gg 1$, most electrons are in the ground state, that is, the state of one-dimensional particle with no $\perp$ motion. Dividing Equation (5.44) by (5.45), we then find that the heat capacity per particle, $c_v = U/N$, approaches $\frac{1}{2}kT$ in the limit $\theta T_0/T \gg 1$, as expected from the law of equipartition which states that each degree of freedom is associated with an energy of $\frac{1}{2}kT$. In this limit P_{xx} and P_{yy} also vanish, to the order $\exp(-\theta T_0/T)$ as expected from the behavior of a one-dimensional gas.

A. DEGENERATE CASE

From the expressions (5.32)–(5.35) the 'equivalent' Fermi energy of the state is μ/a_n. The criteria for degeneracy of the n state is $(\mu/a_n - 1) \gg kT$. As n increases, this inequality becomes weaker and weaker. Therefore, at a given temperature the higher states are always less degenerate.

Similarly (CC68b)

$$C_2(0, \mu) \equiv C_2(\mu) = \tfrac{1}{2}\mu(\mu^2 - 1)^{1/2} - \tfrac{1}{2}\ln[\mu + (\mu^2 - 1)^{1/2}]$$
$$= \tfrac{1}{2}\mu(\mu^2 - 1)^{1/2} - \tfrac{1}{2} C_1(\mu) \tag{5.46}$$

$$C_3(0, \mu) \equiv C_3(\mu) = \tfrac{1}{2}\mu(\mu^2 - 1)^{1/2} + \tfrac{1}{2}\ln[\mu + (\mu^2 - 1)^{1/2}] \tag{5.47}$$

$$C_4(0, \mu) \equiv C_4(\mu) = (\mu^2 - 1)^{1/2}. \tag{5.48}$$

TABLE I

$C_\varkappa(\mu)$.

μ	$C_1(\mu)$	$C_2(\mu)$	$C_3(\mu)$	$C_4(\mu)$
1	0	0	0	0
1.25	0.69315	0.12218	0.81532	0.75000
1.5	0.96242	0.35731	1.31974	1.11803
1.75	1.15881	0.67722	1.83603	1.43614
2.00	1.31696	1.07357	2.39053	1.73215
2.5	1.56680	2.08071	3.64509	2.29129
3.0	1.76275	3.36127	5.12401	2.82843
3.5	1.92485	4.90726	6.83210	3.35410
4.0	2.06344	6.71425	8.77769	3.87298
5.0	2.29243	11.10123	13.39366	4.89898
6.0	2.47789	16.50930	18.98718	5.91608
7.0	2.63392	22.93175	25.56567	6.92820
8.0	2.76866	30.36469	33.13335	7.93725
10.0	2.99322	48.25276	51.24598	9.94987

In Table I the functions C_k are given for $1 \leqslant \mu \leqslant 10$. It is easy to see from the definition of the C_k functions that $C_k(\mu) = 0$ if $\mu < 1$. Therefore the sum in Equations (5.32)–(5.35) terminates at s such that

$$a_s \leq \mu < a_{s+1}\,. \tag{5.49}$$

Physically this means that energy levels up to $n = s$ are occupied and all levels above $n = s + 1$ are vacant. The last level to be occupied is given by the criterion

$$s = \frac{\mu^2 - 1}{2H/H_q} \quad \text{or} \quad \mu = [1 + 2(H/H_q)\, s]^{1/2}\,. \tag{5.50}$$

Because each time when μ exceeds a_s an extra term is added to the sum in the equation of state, discontinuities in the derivatives of thermodynamic variables exist and at $a_s = \mu$, a 'transition' takes place (see Figure 2). These discontinuities are associated with the behavior of the density of states which also shows such discontinuities in the derivatives, as shown in Figure 3.

In particular when $\mu \leqslant (1 + 2H/H_q)^{1/2}$, only the first term in the sum appears and such a gas behaves as a one-dimensional gas. In this case the $\perp$-stress vanishes. At $H/H_q = 1$, the critical density for transition into a one-dimensional gas is approximately 10^6 g/cm^3. Any finite temperature will destroy this one dimensional behavior, however. From the general expressions (5.32)–(5.35), it is easily derived that if degeneracy prevails, the residual $\perp$-pressure is largely due to the state $n = 1$ and is given by

$$P_{xx} = P_{yy} = (2\pi^3)^{-1/2}\, (H/H_q)^2\, (mc^2/\lambda_c^3) \left[\frac{T/T_0}{1 + 2H/H_q}\right]^{1/2} \exp(-\Lambda)$$

$$T\Lambda/T_0 \equiv (1 + 2H/H_q)^{1/2} - \mu\,. \tag{5.51}$$

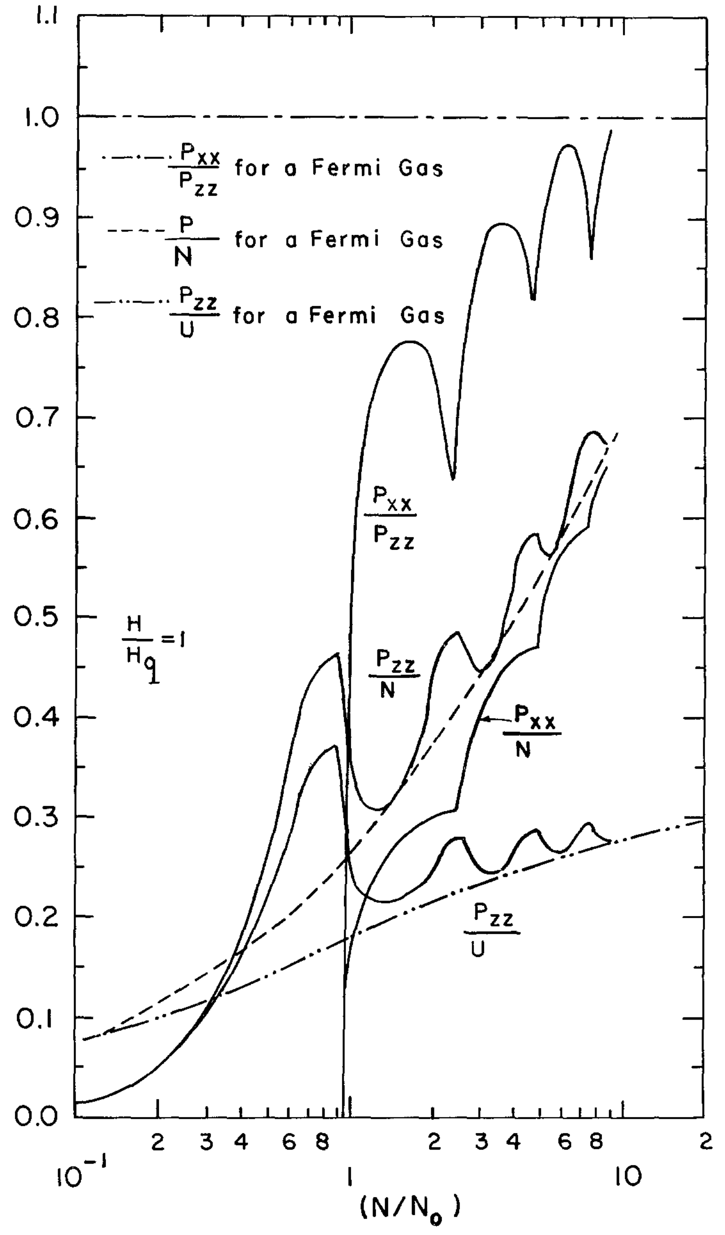

Fig. 2. Functional dependence of P_{xx}/P_{zz}, P_{zz}/N, P_{xx}/N on N/N_0 for the degenerate case at $H/H_q = 1$. The corresponding functions for a Fermi gas are also shown for comparison.

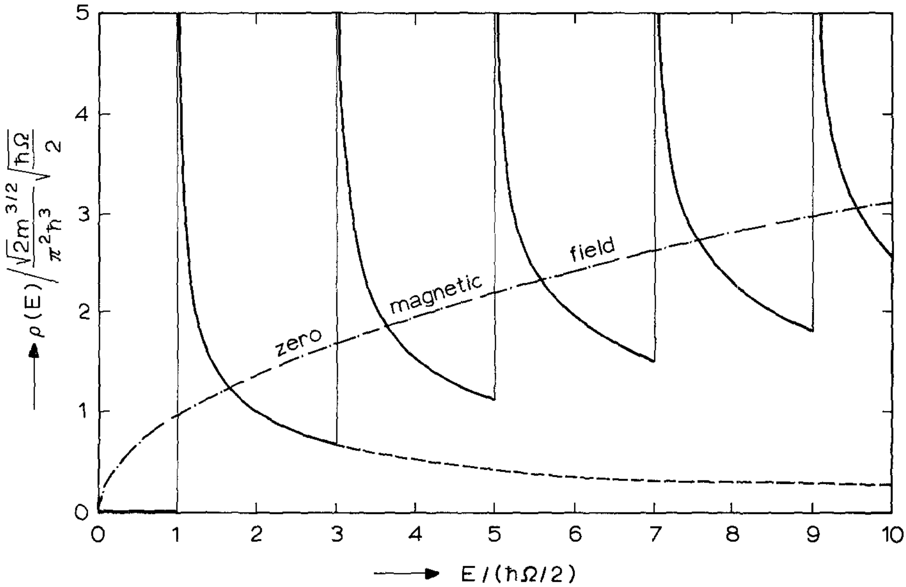

Fig. 3. The oscillating behavior of the density of final states of an electron in a magnetic field (non-relativistic case). [For details on the derivation see KMH65]

6. Radiation Processes in a Magnetic Field

(1) *Synchrotron Radiation.* In a magnetic field an electron can make a transition from one quantized orbit to another, emitting a photon: (if the energy of the photon exceeds $2mc^2$ pairs may be emitted):

$$e^- \to e^- + \gamma . \tag{6.1}$$

Such a process is strictly forbidden in the classical case. This radiation process is called 'synchrotron radiation' because it was first studied in the design of electron synchrotron accelerators. Equation (6.1) remains the limiting factor in designing circular electron accelerators. In nature, synchrotron radiation from energetic electrons in relatively weak fields is the main source of radio emission (and in some cases also of optical and x-ray emissions).

(2) *Bremsstrahlung Process.* An electron collides with an ion and makes a transition from a state n to another state n' ($n=n'$ and $n\neq n'$ are allowed).

$$e_n^- + (Z, A) \to e_{n'}^- + (Z, A) + \gamma . \tag{6.2}$$

This radiation process emits a continuum as in the field free case. The bremsstrahlung process takes place in a dense medium and the effect of the medium on the emission process cannot be fully neglected. This problem is discussed in greater detail in Section 12. The rest of this section will then be devoted to the discussion of the synchrotron radiation process.

A. SYNCHROTRON RADIATION

According to the eigenvalues of electrons in a magnetic field (Equation (3.5)), the frequency of synchrotron radiation, ν, is given by the equation:

$$h\nu = E(p_z, n) - E(p_z, n'). \tag{6.3}$$

Therefore synchrotron radiation in a homogeneous field is emitted in the form of discrete lines, but if the field varies by a factor of two in the emitting region these lines are smeared out into a continuum. However, the lowest frequency of this continuum is given by $n = n' + 1$. If we consider the relativistic case with large values of n then

$$\nu = \frac{E(p_z', n) - E(p_z', n')}{h} \simeq \frac{1}{h} \frac{H/H_q}{E/mc^2} mc^2 = 2.8 \times 10^{10}\, H\gamma^{-1} \quad (\mathrm{Hz}) \tag{6.4}$$

where $\gamma = E/mc^2$ and H is in G.

For example, at $H = 10^6$ G and $\gamma = 10^6$ (10^{12} eV electron) the minimum frequency ν_m of emission is 3×10^{10} Hz and the corresponding wavelength is 1 cm. ν_m increases with decreasing n, electron parallel energy (p_z), and with increasing H. For example, the energy gap between the ground state $n = 0$ and the first excited state $n = 1$ at a field of 10^{12} G is approximately 50 keV.

As discussed earlier, according to the field strength and the electron energy and the rate of emission, synchrotron radiation is studied in two domains:

a. *The Classical Relativistic Domain.*

This is the regime encountered most frequently in astrophysics: a large electron energy ($\gamma \geqslant 10^3$) and a low field ($H \lesssim 10^{-2}$ G). Under this circumstance an electron loses a negligible fraction of its perpendicular kinetic energy in each orbit. That is, the mean lifetime of the electron against losing its kinetic energy, τ, must be much larger than the period $P = \omega^{-1}$. As we will show later, this condition is fulfilled if

$$(e^2/\hbar c)\, \gamma^2 (H/H_q) \ll 1$$

or

$$\gamma^2 H \ll 6 \times 10^{15}. \tag{6.5}$$

For example, at $\gamma = 10^6$ the electron no longer radiates according to the classical theory when $H > 10^3$ G. At $\gamma = 10^8$ the limiting field is only 10 G, and at $H = 10^{-6}$ G (galactic field) an electron no longer radiates classically when $\gamma > 10^{11}$ (10^{17} eV).

When (6.5) is satisfied, the 'radiation reaction' is negligible and the rate of emission is given by classical electrodynamics, using the Lienard-Wiechert potential. The rate of emissions has been discussed extensively previously. For completeness we give the results below. We will first give the general quantum mechanical expression for the spectral distribution of the emitted radiation and then some approximate expressions.

Using the exact wave function of a relativistic electron in a magnetic field, Klepikov (K54) first performed the computation of the intensity in the case where the sum over the initial and final quantum numbers can be approximated with an integral. The result turns out to be (per unit distance)

$$I = \frac{3\alpha}{\pi^2} \frac{mc^2}{\lambda_c} \frac{x}{2+3x} \frac{x^2}{E} \mathcal{M}(x, y) \tag{6.6}$$

$$x = \frac{E}{mc^2} \frac{H}{H_q} \qquad y = \left(\frac{3xE}{2+3x}\right)^{-1} h\nu \qquad \alpha = e^2/\hbar c$$
$$\mathcal{M}(x, y) = \sum_{i=1}^{3} \mathcal{M}_i(h\nu/E)\, J_i(x, y) \tag{6.7}$$

$$\begin{aligned} \mathcal{M}_i(x) &= 1 + (1-x)^{-2} \quad i = 1 \\ &= 2(1-x)^{-1} \quad i = 2 \\ &= [x(1-x)^{-1}]^2 \quad i = 3 \end{aligned} \tag{6.8}$$

$$J_1(xy) = \int_0^\infty \mathrm{d}s \cosh^5 s K_{2/3}^2(t)$$

$$J_2(xy) = \int_0^\infty \mathrm{d}s \cosh^3 s \sinh^2 s K_{1/3}^2(t) \tag{6.9}$$

$$J_3(xy) = \int_0^\infty \mathrm{d}s \cosh^5 s K_{1/3}^2(t), \qquad t = y \cosh^3 s\,[2 + 3x(1-y)]^{-1}.$$

The structure of Equation (6.6) is quite complex and it is hard to compare it with the classical expression

$$I = \frac{\sqrt{3}\alpha}{2\pi} \frac{mc^2}{\lambda_c} \frac{x}{E} k\left(\frac{2}{3x} \frac{h\nu}{E}\right) \tag{6.10}$$

where the function $k(s)$ is defined as

$$k(z) = z \int_z^\infty \mathrm{d}x K_{5/3}(x) \rightarrow \begin{array}{ll} 2.14\, z^{1/3} & z \ll 1 \\ 1.25\, z^{1/2} e^{-z} & z \gg 1 \end{array}. \tag{6.11}$$

($K_n(x)$ is the McDonald function.) Its behavior can be seen in Figure 4.

There is however, one regime, i.e., when

$$h\nu \ll E \tag{6.12}$$

in which the complex mathematical nature of Equation (6.6) can be reduced to the

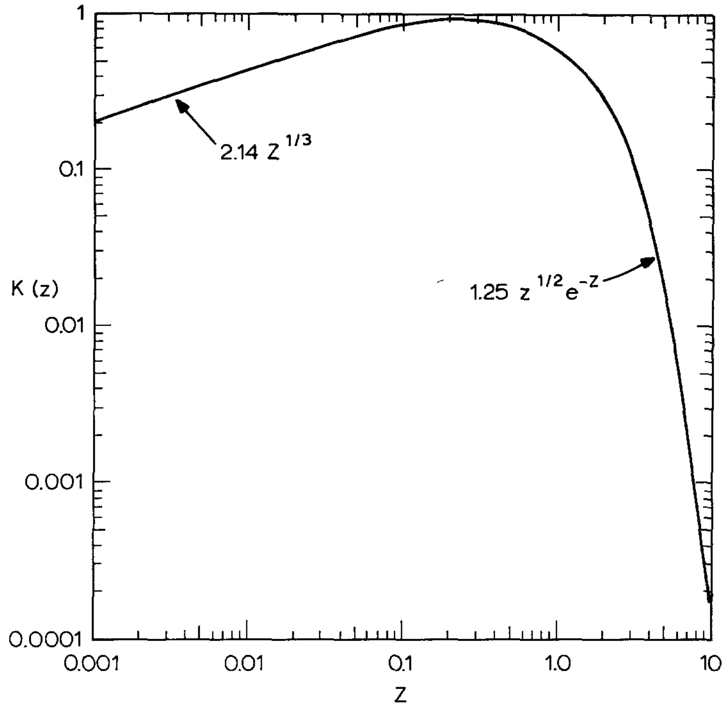

Fig. 4. The bremsstrahlung function $\kappa(z)$, Equation (6.11) of the text.

following simple formula

$$
\begin{aligned}
I &= \frac{\sqrt{3}\alpha}{2\pi}\frac{mc^2}{\lambda_c}\frac{x}{E}(1 - h\nu/E)\,k(2s) \\
s &\equiv \frac{y}{2 + 3x(1-y)} \simeq (3x)^{-1}\frac{h\nu}{E}\left(1 + \frac{h\nu}{E}\right)
\end{aligned}
\tag{6.13}
$$

which is almost identical with the classical expression Equation (6.10). The explicit reduction of Equation (6.6) to Equation (6.13) is done in Erber article (E66). One final remark about Equation (6.6) is that its validity is related to the following inequalities being satisfied:

$$\gamma \equiv E/mc^2 > 1 \qquad \gamma - h\nu/mc^2 \gg 1\,.$$

The total energy dissipated per unit distance Δl was obtained by Klepikov by using the general expressions (6.6). The result is

$$\frac{\Delta E}{\Delta l} = \tfrac{2}{3}\alpha\frac{mc^2}{\lambda_c}\,g(x) \tag{6.14}$$

$$g(x) = \begin{cases} x^2(1 - 5.95x) & x \ll 1 \\ 0.556\,x^{2/3} & x \gg 1\,. \end{cases} \tag{6.15}$$

The total power of emission is given by

$$-\frac{\mathrm{d}E}{\mathrm{d}t} = \tfrac{2}{3}\alpha \frac{mc^2}{\hbar}\left(\frac{H}{H_q}\right)^2 mc^2\,\beta^2\gamma^2 \equiv \Gamma \frac{\beta^2}{1-\beta^2} = \Gamma \frac{E^2 - m^2c^4}{m^2c^4} \tag{6.16}$$

with

$$\Gamma = \tfrac{2}{3}\alpha \frac{mc^2}{\hbar}\left(\frac{H}{H_q}\right)^2 mc^2\,.$$

Integration of Equation (6.16) gives the variation of the energy with time, i.e.,

$$E/mc^2 = \coth\left(\Gamma t/mc^2 + k\right). \tag{6.17}$$

As $t \to \infty$, $E \to mc^2$, as it should because the particle is completely stopped. The value of the constant k is easily found to be

$$k = \tfrac{1}{2}\ln\frac{\gamma_0 + 1}{\gamma_0 - 1} \tag{6.18}$$

with

$$\gamma_0 = (E/mc^2)_{t=0}\,. \tag{6.19}$$

The half-life of the electron, τ, is defined by the equation

$$\coth\left(\Gamma\tau/mc^2 + k\right) = \tfrac{1}{2}\gamma_0$$

and it is given by

$$\tau = \tfrac{1}{2}t_s \ln\left(\frac{\gamma_0 + 2}{\gamma_0 - 2}\,\frac{\gamma_0 - 1}{\gamma_0 + 1}\right), \qquad t_s = \frac{mc^2}{\Gamma} \tag{6.20}$$

For $\gamma_0 \gg 1$

$$\tau \simeq t_s/\gamma_0 \tag{6.21}$$

i.e.,

$$\tau = \left[\frac{2}{3}\,\alpha\,\frac{mc^2}{\hbar}\left(\frac{H}{H_q}\right)^2 \gamma_0\right]^{-1}. \tag{6.22}$$

At $\gamma_0 \simeq 10^6$ and $H = 10^3$ G, the lifetime is only 10^{-5} sec, corresponding to a mean free path of 3×10^{-5} cm. If we now require that τ should be greater than the period ω_L where ω_L is given by Equation (3.3), we easily obtain

$$\frac{e^2}{\hbar c}\,\gamma^2\,\frac{H}{H_q} < 1 \tag{6.23}$$

or

$$\gamma^2 H < 6.10^{15}$$

as discussed previously [see Equation (6.5)].

b. *The Low Quantum Number Region* (CFC69)

When the energy of the electron is small, only the lower quantum states are occupied. The transition between neighboring states are then important. The nature of the synchrotron radiation is very different from that of the case of large quantum number n.

The radiation rate has been computed in analogy with the general case. The transition between two states n and n' gives rise to a photon of energy

$$\hbar\omega = mc^2 \frac{\varepsilon - \cos\theta p_z/mc}{\sin^2\theta}\left[1 - \left\{1 - \frac{2\sin^2\theta\,(n-n')}{(\varepsilon - \cos\theta p_z/mc)^2}\right\}^{1/2}\right] \tag{6.24}$$

where ε is the energy of the initial state:

$$\varepsilon = E/mc^2 = [1 + (p_z/mc)^2 + 2nH/H_q]^{1/2} \tag{6.25}$$

n' is the quantum number of the final state, and θ is the angle of the emitted photon with respect to the axis of the magnetic field.

It may appear that the emission gives rise to a continuum on account of the θ-dependence. In the limit $H/H_q \ll 1$ we can expand Equation (6.24) with the following result

$$\hbar\omega/mc^2 = (n - n')\,H/H_q + \theta\left[(\hbar\omega/mc^2)^2\cos^2\theta\right] \tag{6.26}$$

which corresponds to a narrow line emission of width $\simeq (\hbar\omega/mc^2)^2\,mc^2$. In the case $H/H_q \gg 1$ the line is smeared into a band and the wavelength of the emitted radiation depends on the angle of emission.

The radiation rate (radiation energy per unit volume per time) is

$$I(n) = \frac{e^2c}{\lambda_c^2}\left[\frac{1}{4\pi^2}\frac{H}{H_q}\frac{1}{\lambda_c^3}\int_{-\infty}^{+\infty} \mathrm{d}x f(x)\right]\sum_{n'}\int_0^\pi \sin\theta\,\mathrm{d}\theta\left[1 - f(x')\right] \times \frac{w^2\varepsilon' F(n, n', w, \theta)}{\varepsilon' - (x - w)\cos\theta} \tag{6.27}$$

with

$$x' = x - w\cos\theta\,, \qquad x = p_z/mc\,, \qquad \varepsilon^2 = 1 + x^2 + 2nH/H_q\,. \tag{6.28}$$

$w = \hbar\omega/mc^2$ is given by Equation (6.24). $f(x)$ is the usual Fermi distribution. The function $F(n, n', w, \theta)$ has the following form

$$\begin{aligned} F(n, n', w, \theta) = {} & [\omega_1 I_{n'-1,n}(y) - \omega_2 I_{n',n-1}(y)]^2 \\ & + \cos^2\theta\,[\omega_1 I_{n'-1,n}(y) + \omega_2 I_{n',n-1}(y)]^2 \\ & + \sin^2\theta\,[\omega_3 I_{n'-1,n-1}(y) - \omega_4 I_{n',n}(y)]^2 \\ & - 2\sin\theta\cos\theta\,[\omega_1 I_{n'-1,n}(y) + \omega_2 I_{n',n-1}(y)] \\ & \times [\omega_3 I_{n'-1,n-1}(y) - \omega_4 I_{n',n}(y)] \end{aligned} \tag{6.29}$$

$$y = w^2\sin^2\theta/[4(H/H_q)]$$

where

$$
\begin{aligned}
&\left(\eta^2 \equiv 1 + 2n \frac{H}{H_q}\right) \\
&\omega_1 = \tfrac{1}{4}\left[(1 + s'/\eta')(1 - s/\eta)\right]^{1/2} \\
&\qquad\qquad \times \left\{(1 - w\cos\theta/\varepsilon')^{1/2} + ss'(1 + \cos\theta/\varepsilon')^{1/2}\right\} \\
&\omega_2 = \tfrac{1}{4}\left[(1 - s'/\eta')(1 + s/\eta)\right]^{1/2} \\
&\qquad\qquad \times \left\{ss'(1 - w\cos\theta/\varepsilon')^{1/2} + (1 + w\cos\theta/\varepsilon')^{1/2}\right\} \\
&\omega_3 = \tfrac{1}{4}\left[(1 + s'/\eta')(1 + s/\eta)\right]^{1/2} \\
&\qquad\qquad \times \left\{(1 - w\cos\theta/\varepsilon')^{1/2} - ss'(1 + w\cos\theta/\varepsilon')^{1/2}\right\} \\
&\omega_4 = \tfrac{1}{4}\left[(1 - s'/\eta')(1 - s/\eta)\right]^{1/2} \\
&\qquad\qquad \times \left\{(1 + w\cos\theta/\varepsilon')^{1/2} - ss'(1 - w\cos\theta/\varepsilon')^{1/2}\right\}.
\end{aligned}
\tag{6.30}
$$

The $I_{\alpha\beta}(x)$ functions are defined in the Appendix I.

Equation (6.30) is too complicated to be analyzed in full generality. We will confine ourselves to the non-relativistic case and to the transition $n=1$ to $n'=0$.

In this case the various ω_k simply become

$$\omega_1 = \omega_3 = \omega_4 = 0, \qquad \omega_2^2 = \tfrac{1}{2}H/H_q. \tag{6.31}$$

The function $F(1, 0, w, \theta)$ reduces to

$$F(1, 0, w, \theta) = \tfrac{1}{2}\frac{H}{H_q}(1 + \cos^2\theta)\, I_{0,0}^2 = \tfrac{1}{2}\frac{H}{H_q}(1 + \cos^2\theta). \tag{6.32}$$

For the non-degenerate case we can take $f(x') < 1$. The first parenthesis in Equation (6.27) is therefore seen to be the particle density N_e. (see Equation (5.20)). The final expression is simply

$$I(1, 0, \theta) = \tfrac{1}{2}\alpha\frac{mc^2}{\hbar}\, mc^2\, N_e w^2 \left(\frac{H}{H_q}\right)(1 + \cos^2\theta). \tag{6.33}$$

c. *Other Related Synchrotron Radiation Processes*

In a magnetic field, there are a number of processes involving free photons and free electrons which are normally forbidden. Most of these processes are of theoretical interest but may be important in astrophysics. These processes have been discussed by Erber (E66). They are:

(1) Pair production by a free photon (of energy MeV) in a magnetic field:

$$\gamma \to e^- + e^+ .$$

This process is normally forbidden in the field free case but is allowed here because electrons in Landau states behave kinetically as one-dimensional particles. The lifetime of the process is given by

$$\tau = l/c$$

where (when $H \ll H_q$)

$$l^{-1} = \frac{1}{2} \frac{\alpha}{\lambda\!\!\!^{-}_c} \frac{H}{H_q} T(x), \qquad x \equiv \frac{1}{2} \frac{h\nu}{mc^2} \frac{H}{H_q}$$

with

$$\begin{aligned} T(x) &= 0.60\, x^{-1/3} & x \gg 1 \\ &= 0.46 \exp\left[-(4/3x)\right] & x \ll 1\,. \end{aligned}$$

The maximum of the function $T(x)$ is at $x \simeq 6$ at which $T(6) = 0.1$. At $H = 10^{-8} H_q$ an $h\nu = 6.10^6$ eV, the mean free path l of the photon is $\simeq 1$ cm or $\tau \simeq 10^{-10}$ sec.

(2) *Photon splitting*

$$\gamma \to \gamma + \gamma\,.$$

This process has been computed by Skobov (S58) using Schwinger's Green's function which is valid only for $H \ll H_q$. The exact form of the Green's function is discussed in Appendix II. In this case the attenuation coefficient is computed to be

$$l^{-1} = \frac{5}{3} \frac{1}{(144\pi)^2} \frac{\alpha^3}{\lambda\!\!\!^{-}_c} \frac{h\nu}{mc^2} \left(\frac{H}{H_q}\right)^2.$$

(See however Adler *et al.*, 1970, A70.)

7. Neutrino Processes in Magnetic Fields

It is known that neutrinos can strongly dissipate thermal energy of stars at later stages of stellar evolution (Ch66a, CCFC69, R65). Most neutrino processes in the field free case also operate in the presence of a magnetic field, but in addition, those photon emission processes which are allowed in the presence of a field, can also emit neutrinos via the $(e\nu)(e\nu)$ interaction.

As in all other cases neutrinos are emitted in pairs. This is due to the nature of the $(e\nu)(e\nu)$ interaction. In the following we list a number of examples of neutrino processes in a magnetic field. For a summary see Canuto (C71).

A. Examples of processes allowed in the absence of field and in a field:
  (1) Bremsstrahlung process $e^- + (Z, A) \to e^- + (Z, A) + \nu + \bar{\nu}$
  (2) Photo neutrino process $e^- + \gamma \to e^- + \nu + \bar{\nu}$
  (3) Electron pair annihilation process $e^- + e^+ \to \nu + \bar{\nu}$
  (4) Plasma neutrino process $\gamma \to \nu + \bar{\nu}$
  (although this process in a vacuum without a field is forbidden, it is still allowed in a field. It is analogous to the photon splitting case (see B (2) below).

B. Examples of processes only allowed in the presence of a field.
  (1) Synchrotron process $e^- \to e^- + \nu + \bar{\nu}$
  (2) Photon splitting process $\gamma \to \nu + \bar{\nu}$, $\gamma \to \gamma + \nu + \bar{\nu}$

One thing worth noting in the precence of a field is that cross-section loses its meaning. In a field free case, particles can move freely in any direction and a cross-section can be defined such that a beam of particles of density N travelling with velocity v striking a group of stationary particles of density N_2 has a reaction rate proportional to N, $N_2 v$, where the constant of proportionality is the cross-section σ. In the presence of a field the motions of particles are confined and there is only one velocity component in the usual sense. Therefore cross-section has no physical meaning. On the other hand, the transition probability (which in the field free case is σv) is well defined. In any case, in evaluating the energy loss rate (or other quantities of physical interest) one only needs σv which is equivalent to the transition probability. We will give the results for the process (B.1). The result can be easily modified to compute (A.3). We start with the usual V-A type of interaction

$$S = \int \mathrm{d}^4 x \mathscr{L}(x) = \sum_k \int [\bar{\psi}_e(x)\, O_k \psi_e(x)]\, [\bar{\psi}_\nu(x)\, F_k \psi_\nu(x)]\, \mathrm{d}^4 x$$

with

$$F_k = 2^{-1/2} g_k O_k (1 + \gamma_5) .$$

The index k runs only for vector $O_k = \gamma_\mu$ and axial $O_k = i\gamma_\mu\gamma_5$. The $\psi_e(x)$ are the exact electron wave functions in a magnetic field given in Appendix I. Using the standard field-theoretical method one can then compute the neutrino energy loss

$$-\frac{\mathrm{d}u}{\mathrm{d}t} = \sum_i \sum_f (E_i - E_f)\, f(E_i)\, [1 - f(E_f)]\, W$$

where the transition probability per unit time and volume W is defined as

$$W \equiv \frac{|S|^2}{\Omega T} .$$

A. NEUTRINO SYNCHROTRON ENERGY LOSS

The neutrino luminosity for this process turns out to be (CCCFC70)

$$l = l_0 \frac{H}{H_q} \sum_n \sum_{n'} \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}x' \, [\varepsilon_n(x) - \varepsilon_{n'}(x')] \times f_n(x)\, [1 - f_{n'}(x')] \int_0^1 \mathrm{d}\varrho\, I(x, x', \varrho) \qquad (7.1)$$

where

$$l_0 = \frac{1}{6} \frac{1}{(2\pi)^8} \frac{g^2}{c\hbar^2} \frac{mc^2}{\lambda_c^8} = 1.7475 \times 10^{18} \text{ erg/cc sec} \qquad (7.2)$$

and

$$I(x, x', \varrho) = q_M^4 (A + \varrho B) + q_M^2 (q_3 C + q_0 G)^2$$
$$q_M^2 = [\varepsilon_n(x) - \varepsilon_{n'}(x)]^2 - (x - x')^2 \tag{7.3}$$

$$A \equiv \omega_1^2 \phi_1^2 + \omega_2^2 \phi_2^2 - 4\omega_3\omega_4\phi_3\phi_4$$
$$B \equiv \omega_1^2 \phi_1^2 + \omega_2^2 \phi_2^2 + 4\omega_3\omega_4\phi_3\phi_4$$
$$C \equiv \omega_3\phi_3 - \omega_4\phi_4 \tag{7.4}$$
$$D \equiv \omega_3\phi_3 + \omega_4\phi_4$$

$$\phi_1 = \Phi(n|n'-1) \qquad \phi_2 = \Phi(n-1|n')$$
$$\phi_3 = \Phi(n|n') \qquad \phi_4 = \Phi(n-1|n'-1)$$

$$\Phi(n|n') = (n!\,n'!)^{-1/2} e^{-t/2} t^{n+n'/2} {}_2F_0(-n', -n; -t^{-1})$$
$$t = (2H/H_q)^{-1} q_M^2 (1-\varrho). \tag{7.5}$$

A numerical integration of Equation (7.1) was performed for $H/H_q = 1$ and four different temperatures. The results are reported in Table II and III. They are usually larger by a factor of $\sim 10^2$ from the Landstreet results (L66) which were obtained

TABLE II

ϱ_6 (g/cm³)		I(erg/cm³ sec)
	$T = 5.9303 \times 10^7$ K	
3.7146		0.26037×10^{-1}
5.598		0.4998
5.8417×10		4.054×10^5
2.0736×10^2		1.9397×10^7
	$T = 3.7418 \times 10^8$ K	
4.802×10^{-1}		1.2063×10^{12}
1.44		5.092×10^{12}
3.7146		2.107×10^{13}
5.598		3.7116×10^{13}
5.8417×10		1.2984×10^{14}
2.0736×10^2		2.5338×10^{13}

TABLE III

ϱ_6 (gr/cm³)		I(erg/cm³ sec)
	$T = 5 \times 10^8$ K	
6.7082×10^{-2}		1.64071×10^{13}
3.7146		1.7649×10^{14}
5.598		2.675×10^{14}
5.8417×10		7.37×10^{14}
	$T = 9.3988 \times 10^8$ K	
6.7082×10^{-2}		1.659×10^{15}
3.7146		8.51×10^{15}
5.598		1.0586×10^{16}
5.8417×10		3.1804×10^{16}

after integrating over all the quantum numbers n and n'; this procedure is valid only when the problem is quasi-classical, i.e., when the density is much higher than ϱ_6; this is surely not the case considered in the numerical analysis performed in (CCCFC69). Therefore the comparison with Landstreet results cannot be taken too seriously. In Figure 5 we report the region in the ϱ–T plane where the neutrino synchrotron process is important.

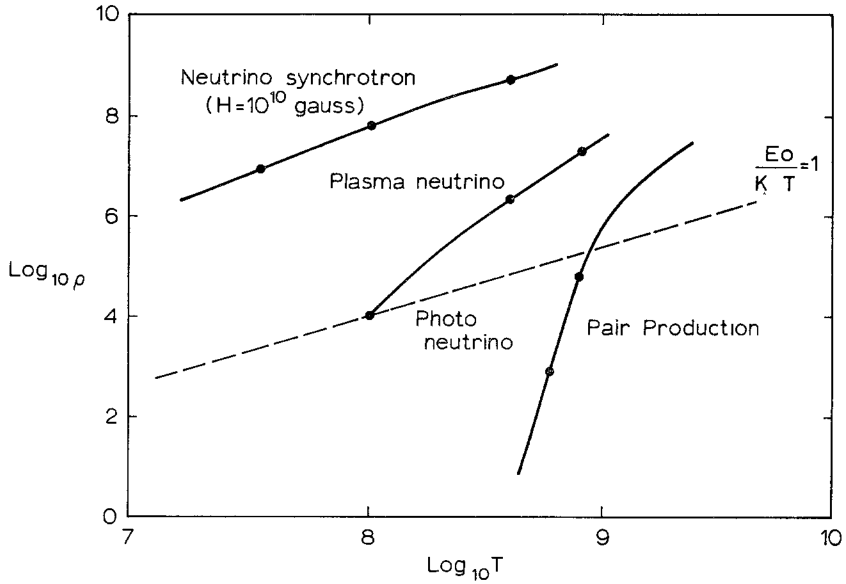

Fig. 5. Region describing the relative importance of the neutrino synchroton, plasma neutrinos, photoneutrinos, and neutrino pair processes. (L66)

B. PLASMAN NEUTRINOS (CCC70)

As seen in Figure 5 the plasmon neutrino process

$$\gamma \to e^- + e^+ \to \nu + \bar{\nu}$$

is of primary importance at relatively low temperature and high density. This process has been repeated in the presence of a strong magnetic field (CCC70). For the transverse decaying photon the neutrinos energy loss is given by

$$Q(\theta = 0) = \bar{Q}\omega_p^4 \int_{\omega_0}^{\infty} \mathrm{d}\omega \, \frac{\omega^6}{(\omega \pm \omega_c)^2} N_l [1 - N_l^2] \, f(\omega) \tag{7.6}$$

where

$$N_l^2 = 1 - \frac{\omega_p^2}{\omega^2} \frac{\omega}{\omega \pm \omega_c}, \quad \omega_p^2 = \frac{4\pi e^2 N_e}{m}, \quad \omega_c = \frac{eH}{mc} \tag{7.7}$$

$$f(\omega) = \{\exp(\hbar\omega/kT) - 1\}^{-1} \tag{7.8}$$

$$\bar{Q} = \frac{1}{12\alpha}\frac{1}{(2\pi)^5}\frac{g^2}{\hbar c^q}\left(\frac{\text{erg}}{\text{cc sec ster}}\right). \tag{7.9}$$

The lower limit ω_0 has to be chosen such that $N_e^2 < 1$. Equation (7.6) is for propagation along the magnetic field, $\theta = 0$. The plus or minus ($l = 1, 2$) in the refractive index refer to ordinary (0) and extraordinary (X) waves.

An analogous computation gives for $\theta = \pi/2$

$$Q_0(\theta = \pi/2) = \bar{Q}\omega_p^4 \int_{\omega_p}^{\infty} d\omega\, \omega^4 N_0 (1 - N_0^2) f(\omega) \tag{7.10}$$

$$N_0^2 = 1 - \omega_p^2/\omega^2 \tag{7.11}$$

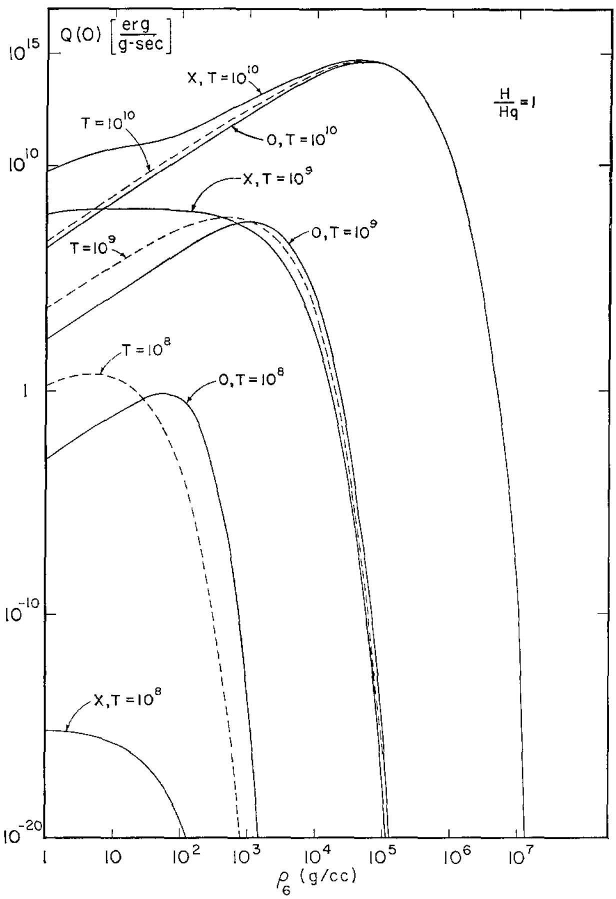

Fig. 6. Energy loss per unit mass and unit solid angle vs ϱ_6 and different temperatures for $H = H_q$ and $\theta = 0$. As explained in the text the symbols O and X stand for ordinary and extra-ordinary modes. The dashed lines correspond to $H = O$.

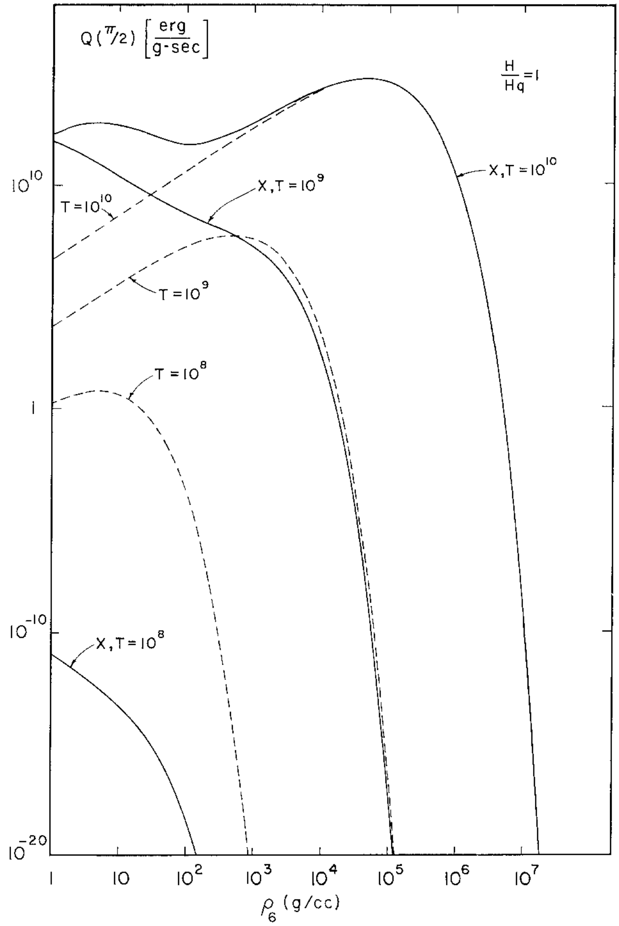

Fig. 7. The same as in Figure 6 for $\theta = \pi/2$. The ordinary mode coincides with the $H = O$-case.

$$Q_x(\theta = \pi/2) = \bar{Q}\omega_p^4 \int_{\omega_0}^{\infty} \mathrm{d}\omega \, \frac{\omega^8}{(\omega^2 - \omega_c^2)^2} N_x [1 - N_x^2] \times \left[1 + (1 - N_x^2)\left(\frac{\omega_c}{\omega}\right)^2\right] f(\omega) \tag{7.12}$$

$$N_x^2 = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_p^2 - \omega_c^2}. \tag{7.13}$$

Equations (7.6), (7.10) and (7.12) were solved numerically for $H = H_q$ for various densities and temperatures. The results are shown in Figures 6 and 7. The general conclusion is that a sizable effect can be found only for low densities and very high field, a situation not met in white dwarf or neutron star interiors. A different situation is encountered when one considers the longitudinal plasmon.

At $\theta = 0$ of the two possible dispersion relations

$$\begin{aligned} \omega^2 &= \omega_p^2 \\ \omega^2 &= \omega_c^2 \end{aligned} \tag{7.14}$$

the first does not depend on the magnetic field while the second is purely magnetic field dependent and correspondent neutrino luminosity is given by

$$Q\,(\theta = 0) = \bar{Q}\omega_p^4\,(\omega_c/\mu)^5\, f\,(\omega_c/\mu)\,. \tag{7.15}$$

At $\theta = \pi/2$ only one dispersion relation is important

$$\omega^2 = \omega_h^2 = \omega_c^2 + \omega_p^2 \tag{7.16}$$

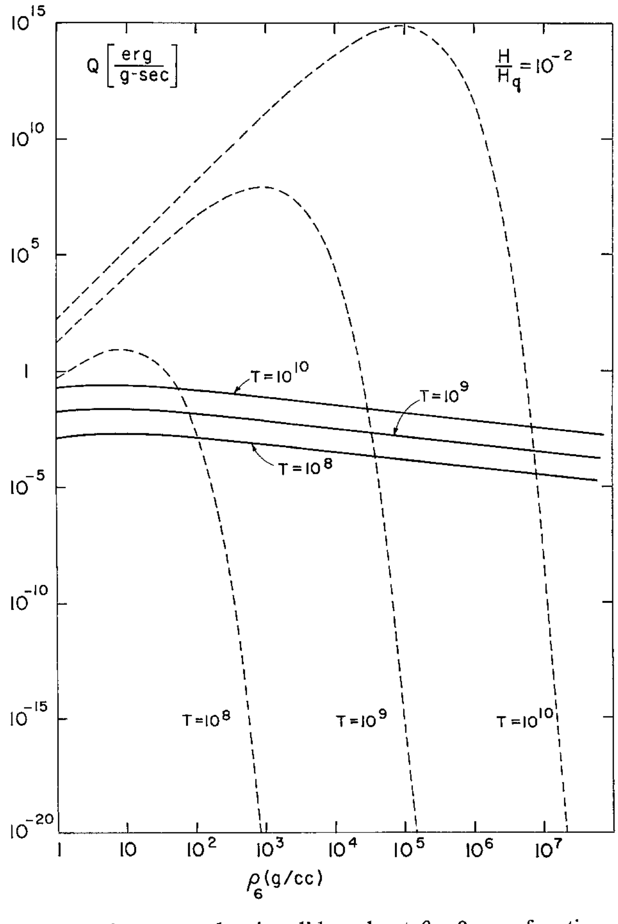

Fig. 8. Energy loss per unit mass and unit solid angle at $\theta = 0$ as a function of ϱ_6 and different temperatures. The solid curve refers to the mode $\omega = \omega_c$ and $H/H_q = 1\bar{0}^2$. The dashed curve refers to the mode $\omega = \omega_p$ which is independent of the field.

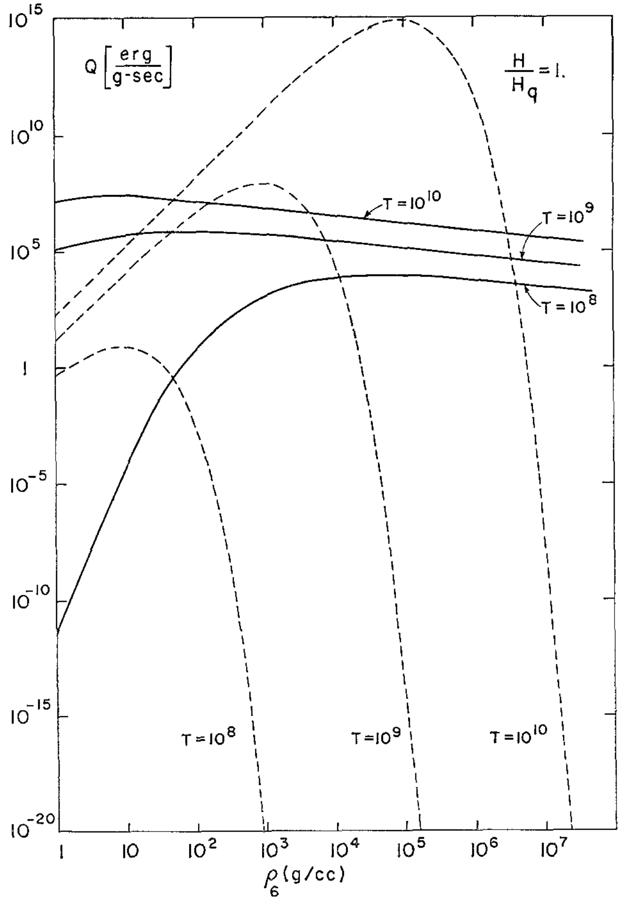

Fig. 9. Same as in Figure 8 for $H=H_q$.

since the other gives

$$\omega^2 = 0 .$$

The neutrinos luminosity turns out to be

$$Q(\theta = \pi/2) = \bar{Q}\omega_p^2\omega_h^5(\omega_p^2 + \tfrac{7}{4}\omega_c^2) f(\omega_h) . \tag{7.17}$$

Since $\omega_p \gg \omega_c$, $\omega_h \approx \omega_p$ and therefore no great difference is expected from the case with $H=0$, at least at $\theta=\pi/2$. In Figures 8 and 9 we reproduce the neutrinos luminosities (Equation 7.14) for $H=10^{-2} H_q$ and $H=H_q$. It is important to note that this magnetic field dependent mode survives at those densities at which the free field case is exceedingly small.

8. Neutron Beta Decay

Because of the Landau levels the electron final states are strongly modified. Strictly speaking, all states of the neutron and the proton are also affected by a field. However, the effect of the field is proportional to m^2. For fields $H \ll 10^{19}$ G, the states of the proton and neutron are not affected.

The neutron mean life τ in a magnetic field has been considered, but the most complete work is due to Fassio-Canuto (FC69). She considered two cases: vacuum and a highly degenerate magnetized state. The general result is that the neutron mean life in a magnetic field begins to be significantly decreased when the field strength is greater than 10^{10} G. The physical reason for the decreasing of the lifetime lies in the fact that the phase space for a one-dimensional particle is dp_z (instead of $p^2\, dp$) and

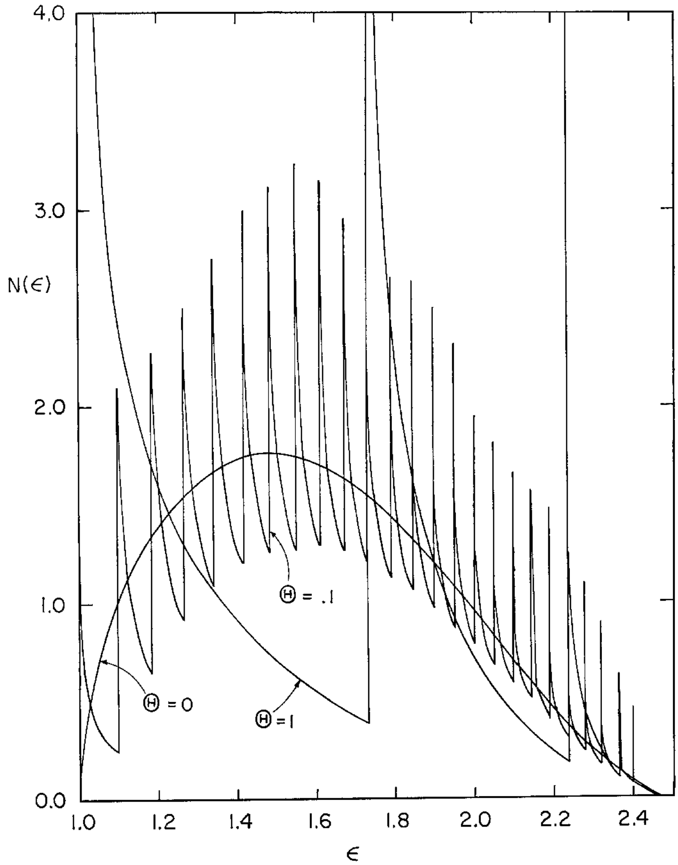

Fig. 10. The β-ray spectrum $N(\varepsilon)$ for $\Theta = 0, 0.1, 1$ with $\Theta \equiv H/H_q$, $H_q = 4.414 \times 10^{13}$ G.

does not vanish at zero electron energy. This is seen in Figure 10 where the differential spectrum, $N(\varepsilon)\, d\varepsilon$ is shown with and without magnetic field. The Fermi type spectrum vanishes at $p = 0$, whereas the one-dimensional magnetized electron does not. This increases the space phase and as a consequence decreases the life time. The discontinuities are due to the density of final states (Figure 3).

The expression for the neutron decay half-life τ is

$$\tau^{-1} = (\bar{\tau})^{-1} \frac{H}{H_q} \sum_{n=0}^{N} \int_{a_n}^{\Delta} (1 - \tfrac{1}{2}\delta_{n0})\, \mathrm{d}\varepsilon \left[1 - f(\varepsilon)\right] \frac{\varepsilon(\Delta - \varepsilon)^2}{(\varepsilon^2 - a_n^2)^{1/2}} \tag{8.1}$$

where

$$a_n^2 = 1 + 2nH/H_q \qquad \Delta = m^{-1}(M_n - M_p)$$
$$(\bar{\tau})^{-1} = g_V^2 (1 + 3\lambda^2) \frac{m^5 c^4}{4\pi^3 \hbar^7} \tag{8.2}$$

where $\lambda \equiv g_A/g_V$ is the ratio of the axial and vector coupling constant and N must be chosen in such a way that $\varepsilon^2 - a_N^2 > 0$. In the nondegenerate case $f(\varepsilon) \ll 1$. The integral can be exactly integrated and we have, with $x \equiv \Delta/a_n$

$$\begin{aligned} \tau^{-1} &= (\bar{\tau})^{-1} \frac{H}{H_q} \sum_{n=0}^{N} (1 - \tfrac{1}{2}\delta_{n0})\, a_n^3 \\ &\quad \times \left[\tfrac{1}{3}(x^2 - 1)^{1/2}(2 + x^2) - x \ln\left(x + \sqrt{x^2 - 1}\right)\right] \\ N &= (\Delta^2 - 1)/(2H/H_q). \end{aligned} \tag{8.3}$$

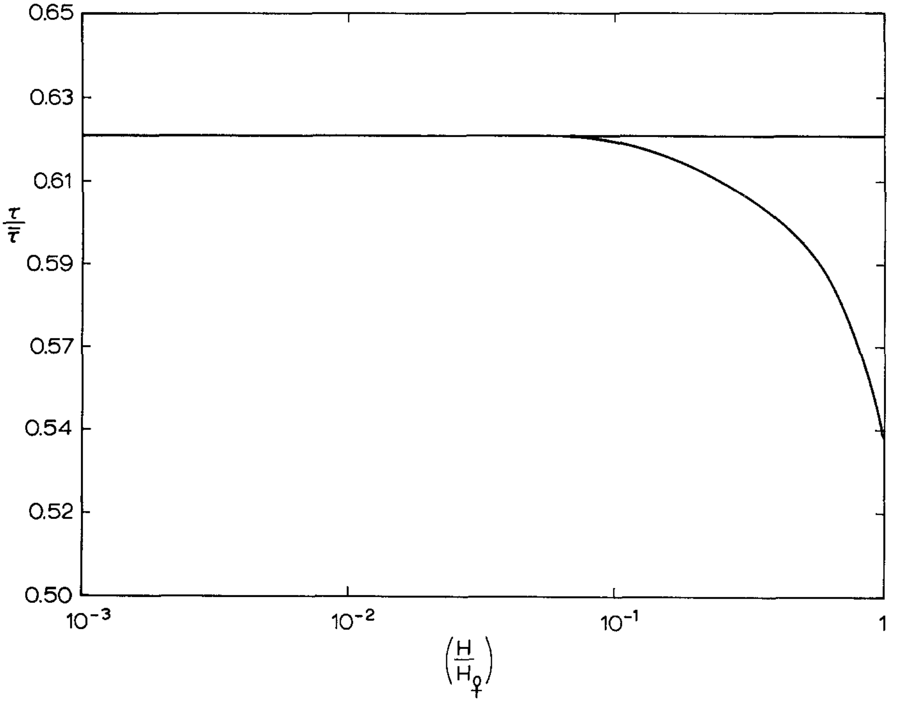

Fig. 11. The neutron mean life in vacuum normalized to a constant factor $\bar{\tau}$, Equation (8.3), is plotted as a function of the magnetic field. The straight line is the neutron mean life when $H = 0$.

When

$$\Delta^2 < 1 + 2H/H_q$$

only the first term contributes and the result is easily seen to be

$$\tau^{-1} = 1.3\,(\bar{\tau})^{-1}\,\frac{H}{H_q}. \tag{8.4}$$

At higher fields $\tau \sim H^{-1}$. The lifetime τ (Equation 8.3) has been computed numerically as a function of H/H_q. Its behavior is shown in Figure 11 where we also give the free neutron lifetime. As said before, the effect of the magnetic field becomes appreciable only at fields greater than 10^{10} G. As the density increases degeneracy appears. In the completely degenerate case we have

$$1 - f(\varepsilon) = H(\varepsilon - \mu) \tag{8.5}$$

where H is the Heaviside step function. As a consequence the lower limit in Equation

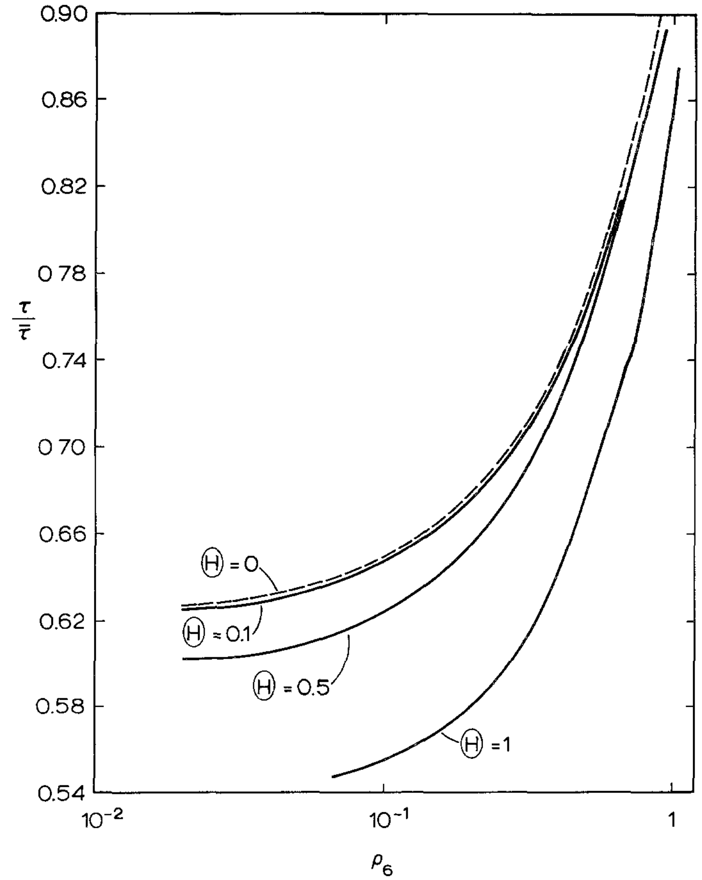

Fig. 12. The neutron mean life normalized as in Figure 11 is plotted vs $\varrho_6 \equiv 10^{-6}\,\varrho/\mu_e$ (ϱ in g/cc) for different values of $\Theta = H/H_q$.

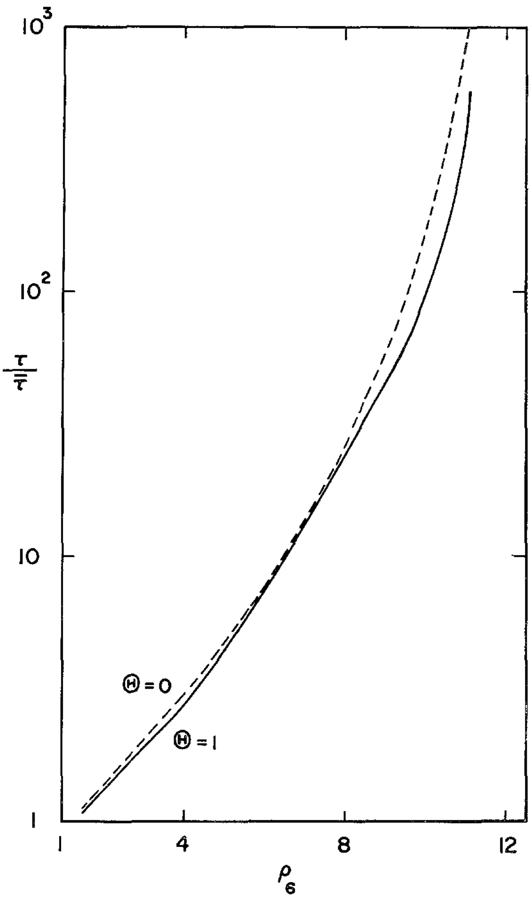

Fig. 13. The same as in Figure 12. At $\varrho \simeq \varrho^* \simeq 10^7$, the neutron mean life becomes infinite, i.e., the neutron becomes a stable particle.

(8.1) is changed to μ. The result is

$$\frac{\bar{\tau}}{(\tau)_{\text{Deg}}} = \frac{\dot{\tau}}{\tau} - \frac{H}{H_q} \sum_{n=0}^{N} \left(1 - \tfrac{1}{2}\delta_{n0}\right) a_n^3 \{(y^2 - 1)^{1/2} \times [x(x - y) + \tfrac{1}{3}(2 + y^2)] - x \ln[y + (y^2 - 1)^{1/2}]\} \tag{8.6}$$

with

$$y \equiv \mu / a_n \qquad N \equiv \frac{\mu^2 - 1}{2H/H_q}. \tag{8.7}$$

In Figure 12, and 13 the lifetime is shown as a function of the density for some values of H/H_q. As the density approaches the values ϱ^* at which the Fermi energy is equal to Δ, the neutrons become stable and the lifetime is infinite. As in a vacuum, the effect of the magnetic field is that of lowering the neutron lifetime. Some astrophysical consequences of this shortening have recently been considered (G69).

9. Dielectric Tensor for a Quantum Plasma

The dispersive character of a quantum plasma in a magnetic field has been the subject of numerous papers and an exhaustive list of references is found in a recent article by Green *et al.* (Gr69). The usual classical analysis is based on the solution of the Boltzmann-Vlasov equation for the distribution function. As discussed at length in Kelly's paper (Ke64) there is no unique way to treat the problem on a quantum mechanical basis. The reason is simply because no precise quantum counterpart of the classical distribution exists. An improved fully quantum-mechanical derivation has recently been achieved and studied in detail (Canuto and Ventura; CV71). The method employed by Kelly is based on the use of the Wigner function [BC62]

$$f(r, p, t) = (\pi\hbar)^{-3} \int \exp(2i\mathbf{p}\cdot\boldsymbol{\lambda})\, \varrho(\mathbf{r} - \boldsymbol{\lambda}, \mathbf{r} + \boldsymbol{\lambda})\, \mathrm{d}^3\lambda \tag{9.1}$$

where $\varrho(r, r', t)$ denotes the single particle density matrix. The Wigner function $f(r, p, t)$ is shown to satisfy Boltzmann-Vlasov equation

$$f(r, p, t) = f_0(p) + f_1(r, p, t) \tag{9.2}$$

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\cdot\nabla f_1 + (e/c)(\mathbf{v} \times \mathbf{B})\nabla_p f_1 = -e[E_1 + (1/c)(\mathbf{v} \times \mathbf{B}_1)]\nabla_p f_0 \tag{9.3}$$

with

$$\begin{aligned} \mathbf{E}_1 &= -\frac{1}{c}\frac{\partial \mathbf{A}_1}{\partial t} - \nabla\phi \\ \mathbf{B}_1 &= \nabla \times \mathbf{A}_1 \end{aligned} \tag{9.4}$$

where $\phi(r, t)$ and $A_1(r, t)$ are the scalar and vector components of the self-consistent electromagnetic field felt by each electron. Introducing the Fourier transform of $f_1(r, t)$ and $E_1(r, t)$ and using standard plasma physics techniques the general expression of the dielectric tensor turns out to be

$$\varepsilon_{\alpha\beta} = (1 - \omega_p^2/\omega^2)\,\delta_{\alpha\beta} - (\omega_p^2/\omega^2)\, n_{\alpha\beta} \tag{9.5}$$

with (we only quote the components of $n_{\alpha\beta}$ we shall need)

$$\begin{aligned} n_{11} &= S[(nJ_n/b)^2\, p_\perp \{v_\perp q_\parallel f_\parallel + n\omega_c f_\perp\}] \\ n_{12} &= -n_{21} = S[i(nJ_n J'_{n'}/b)\, p_\perp \{v_\perp q_\parallel f_\parallel + n\omega_c f_\perp\}] \\ n_{22} &= S[(J'_n)^2\, p_\perp \{v_\perp q_\parallel f_\parallel + n\omega_c f_\perp\}] \\ n_{33} &= S[J_n^2 p_\parallel \{v_\parallel q_\parallel f_\parallel + v_\perp^{-1} v_\parallel n\omega_c f_\perp\}] \end{aligned} \tag{9.6}$$

where

$$\begin{gathered} S[x] \equiv \sum_{n=-\infty}^{+\infty} \int \frac{\mathrm{d}^3 p\,[x]}{\omega + q_\parallel v_\parallel + n\omega_c}, \quad \omega_c = eH/mc, \quad \omega_p^2 = 4\pi e n_e/m \\ f_\parallel = \partial f_0/\partial p_\parallel \qquad f_\perp = \partial f_0/\partial p_\perp \qquad b = q_\perp p_\perp/m\omega_c \end{gathered} \tag{9.7}$$

J_k is the Bessel function of order i. As customary in plasma physics, the wave vector q is taken as $\mathbf{q} = (q_\perp, 0, q_\parallel)$; on the other hand, the momentum variable is taken as $\mathbf{p} = (p_\perp \cos\phi, p_\perp \sin\phi, p_\parallel)$. Having obtained the tensor $\varepsilon_{\alpha\beta}$ the dispersion relation is found by solving the usual equation

$$\det\left[c^2 (q^2 \delta_{\alpha\beta} - q_\alpha q_\beta) - \omega^2 \varepsilon_{\alpha\beta}\right] = 0. \tag{9.8}$$

In order to evaluate the tensor $n_{\alpha\beta}$ one still needs the equilibrium distribution $f_0(p)$ given by Equation (9.1) where the equilibrium density matrix

$$\varrho(\mathbf{r}, \mathbf{r}') = \sum_j w_j \phi_j^* (\mathbf{r}') \, \phi_j (\mathbf{r}) \tag{9.9}$$

with

$$w_j = \{1 + \exp[\tilde{\beta}(E_j - \tilde{\mu})]\}^{-1}$$

requires the use of single particle wave-function of an electron in a uniform magnetic field. For the Maxwell-Boltzmann case, w_j is simply given by $\exp[-\tilde{\beta}(E_j - \tilde{\mu})]$. Using the wave functions given in Appendix I, Kelly's result is the following

$$\begin{aligned} f_0(p_\perp, p_\parallel) &\equiv \frac{\operatorname{tgh}\theta}{m\hbar\omega_c} \left(\frac{\beta}{2m\pi^3}\right)^{1/2} \exp\left[-\frac{\beta p_\parallel^2}{2m} - \operatorname{tgh}\theta \, \frac{p_\perp^2}{m\hbar\omega_c}\right] \\ \theta &\equiv \tfrac{1}{2}\beta\hbar\omega_c \end{aligned} \tag{9.10}$$

for the Maxwell-Boltzmann distribution and $[w^2 \equiv (p_\parallel^2 + p_\perp^2)/m\hbar w_c]$

$$\begin{aligned} f_0(p_\perp, p_\parallel) &= \frac{2e^{-w^2}}{N_e (2\pi\hbar)^3} \sum_{n=0}^{\infty} \sum_{s=\pm 1/2} \\ &\quad \times \frac{(-)^n L_n(2w^2)}{1 + \exp[\beta p_\parallel^2/2M + \beta\hbar\omega_c (n + s + \frac{1}{2}) - \beta\mu]} \end{aligned} \tag{9.11}$$

for Fermi-Dirac statistics. Both $f_0(p_\parallel, p_\perp)$ are normalized in such a way that

$$\int f_0(p_\parallel, p_\perp) \, \mathrm{d}^3 p = 1.$$

With these formulae we are now able to study the propagation parallel and perpendicular to the magnetic field. For the first case $q_\perp = 0$, $q_\parallel = q$ and the result turns out to be

$$\begin{aligned} n_{11} &= n_{22} = n_+ + n_- \\ n_{12} &= -n_{21} = i(n_+ - n_-) \end{aligned} \tag{9.12}$$

$$
\begin{aligned}
2n_{\pm}^{MB} &= \frac{\mp\omega_c}{\omega\pm\omega_c} + \frac{q^2}{m\beta(\omega\pm\omega_c)^2}\left\{\theta\coth\theta \mp \frac{\omega_c}{\omega\pm\omega_c}\right\} \\
2n_{\pm}^{FD} &= \left(\frac{q^2\hbar\omega_c}{2m}\sum_{n=0}^{n_F}\frac{\alpha_n V_n}{(\omega\pm\omega_c)^2 - q^2V_n^2}\right. \\
&\qquad \left.\mp\frac{\omega_c}{q}\sum_{n=0}^{n_F}\alpha_n \ln\left|\frac{\omega\pm\omega_c+qV_n}{\omega\pm\omega_c-qV_n}\right|\right)\Big/\sum_{n=0}^{n_F}\alpha_n V_n \\
n_{33}^{MB} &= 3q^2/m\beta\omega^2 \\
n_{33}^{FD} &= (q^2\mu/m\omega^2)\,\chi(\hbar\omega_c/\mu)
\end{aligned}
\tag{9.13}
$$

with

$$
\begin{aligned}
\chi(s) &= \sum_{n=0}^{n_F}\alpha_n(1-sn)^{3/2}\Big/\sum_{n=0}^{n_F}\alpha_n(1-sn)^{1/2} \\
V_n^2 &= \frac{2}{m}(\mu - n\hbar\omega_c), \qquad \alpha_n = 2\left(\perp - \tfrac{1}{2}\delta_{n0}\right).
\end{aligned}
\tag{9.14}
$$

The dispersion relation (9.5) becomes

$$
\begin{aligned}
\omega^2 &= \omega_p^2(1+n_{33}) \\
\omega^2 &= q^2c^2 + \omega_p^2(1+2n_{\pm})
\end{aligned}
\tag{9.15}
$$

for longitudinal and circularly polarized waves. For waves propagating perpendicular to the magnetic field $q_\perp = q$, $q_\parallel = 0$ the results are

$$
\begin{aligned}
\varepsilon_{11} &= 1 - \frac{\omega_p^2}{\omega^2-\omega_c^2} + \frac{q^2\hbar\omega_c}{2m}\frac{\omega_p^2}{\omega^2}(\varDelta_1 - 4\varDelta_2)\coth\theta \\
\varepsilon_{12} &= -\varepsilon_{21} = \frac{i\omega\omega_c}{\omega^2-\omega_c^2}\frac{\omega_p^2}{\omega^2} - \frac{iq^2\hbar\omega_c}{m}\frac{\omega_p^2}{\omega^2}(\varDelta_1-\varDelta_2)\coth\theta \\
\varepsilon_{22} &= 1 - \frac{\omega_p^2}{\omega^2-\omega_c^2} + \frac{q^2\hbar\omega_c}{2m}\frac{\omega_p^2}{\omega^2}(3\varDelta_1 - 4\varDelta_2)\coth\theta \\
\varepsilon_{33} &= 1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{q^2}{m\beta[\omega^2-\omega_c^2]}\right), \qquad \varDelta_n = (\omega^2 - n^2\omega_c^2)^{-1}
\end{aligned}
\tag{9.16}
$$

for a Boltzmann gas. For a Fermi gas at $T = 0\,\mathrm{K}$, the corresponding dielectric tensor is

$$
\begin{aligned}
\varepsilon_{11} &= 1 - \frac{\omega_p^2}{\omega^2-\omega_c^2} + \frac{q^2\hbar\omega_c}{2m}\frac{\omega_p^2}{\omega^2}(\varDelta_1 - 4\varDelta_2)\,\Psi(\hbar\omega_c/\mu) \\
\varepsilon_{12} &= -\varepsilon_{21} = \frac{i\omega\omega_c}{\omega^2-\omega_c^2}\frac{\omega_p^2}{\omega^2} - \frac{iq^2\hbar\omega_c}{m}\frac{\omega_p^2}{\omega^2}(\varDelta_1-\varDelta_2)\,\Psi(\hbar\omega_c/\mu) \\
\varepsilon_{22} &= 1 - \frac{\omega_p^2}{\omega^2-\omega_c^2} + \frac{q^2\hbar\omega_c}{2m}\frac{\omega_p^2}{\omega^2}(3\varDelta_1 - 4\varDelta_2)\,\Psi(\hbar\omega_c/\mu)
\end{aligned}
\tag{9.17}
$$

$$\varepsilon_{33} = 1 - \frac{\omega_p^2}{\omega^2}\left[1 + \frac{2q^2\mu}{3m(\omega^2 - \omega_c^2)}\chi(\hbar\omega_c/\mu)\right]$$

$$\Psi(s) = 2\sum_{n=0}^{n_F}\alpha_n n(1 - sn)^{1/2} / \sum_{n=0}^{n_F}\alpha_n(1 - sn)^{1/2}. \tag{9.17}$$

10. Transport Processes – Electron Conduction

In general, transport processes may be classified into 2 categories: energy transport and material transport. Generally speaking, material transport often involves energy transport. The former includes radiative transport of energy and electron conduction and the latter includes convection and diffusion.

While all these processes have been studied in the field free case, with certain exceptions (in the non-relativistic low temperature case for ordinary solids) few of these processes have been studied in the case of strong fields. The problem of radiative transport has not been studied at all in the case of strong fields.

The computation of the electrical conductivity is extremely important by itself because problems related to the decay in time of a magnetic field superimposed on a plasma depends on it. Using Maxwell's equation and Ohm's law, one can easily prove that the diffusion equation for a magnetic field H is (J62)

$$\frac{\partial H}{\partial t} = -\left(\frac{c^2}{4\pi\sigma}\right)\nabla^2 H \tag{10.1}$$

where σ is the electrical conductivity. Approximating $\nabla^2 H$ by HL^{-2} where H is the field and L is the dimension of the field. Equation (10.1) yields an exponential solution for the field H, i.e.,

$$H(t) = H(0)\exp\left(-\frac{c^2}{4\pi\sigma L^2}t\right)$$

which shows that the initial configuration of H will decay in a diffusion time τ given by

$$\tau = \frac{4\pi\sigma L^2}{c^2}.$$

The problem of electron conduction in the absence of a field has been studied since 1932, and the most recent computation has been given by Hubbard and Lampe (HL69) and by Canuto (C69) for the non-relativistic and relativistic case respectively. These works include the ion-ion correlation to eliminate the forward divergence problem of the Coulomb scattering cross-section. The results of Hubbard-Lampe and Canuto are believed to be, up to the time of this article, the best electron conductivity in the absence of a field. The results reported below will be compared to the values of these authors.

In classical theory of conductivity it has been shown that the ratio of thermal conductivity to electrical conductivity is given by the Wiedeman-Franz law, which states that

$$\sigma_{\text{th}}/\sigma_0 = \frac{\pi^2}{3}\frac{k^2}{e^2}T$$

where σth is the thermal conductivity and σ is the electrical conductivity defined (C69) ($\varrho_6 \equiv \varrho \times 10^{-6}$, ϱ in g/cc)

$$\begin{aligned}\sigma_0 &= \bar{\sigma}_0 G(\varrho_6^{2/3})\\ \bar{\sigma}_0 &= \tfrac{4}{3}\left[(2\pi)^3 \lambda_c^3 Z \mathcal{N}_i\right]^{-1} (\alpha Z\hbar/mc^2)^{-1}\end{aligned} \tag{10.2}$$

$$G(x) = x(1+x)^{-1/2}\left\{\int_0^\pi d\theta \sin\theta(1-\cos\theta)\left[2+x(1+\cos\theta)\right]\right.$$
$$\left.\times\left[x(1-\cos\theta)+\frac{2\alpha}{\pi}x^{1/2}(1+x)^{1/2}\right]^{-2}\phi(x,\theta)\right\}^{-1} \tag{10.3}$$

$$\phi(x,\theta) = 1 + 3\int_0^\infty ds\,(s\xi)^{-1}\sin(s\xi)\,g(s) \tag{10.4}$$

$$\begin{aligned}&\xi = 2.69\,\mu_i^{1/3}\mu_e^{-1/3}(1-\cos\theta)^{1/2}, \qquad \mathcal{N}_i = \frac{N_{\text{ions}}}{\Omega}\\ &\mu_i = A, \qquad \mu_e^{-1} = Z/A.\end{aligned} \tag{10.5}$$

The function $g(x)$ is the pair-correlation function which takes into account the ion-ion interactions. Its value depends on a parameter $\Gamma^{-1} = (z^2e^2/KT)\left[(4\pi/3)(\mathcal{N}_i/\Omega)\right]^{1/3}$ which measures the strength of the ion-ion interaction. For high values of $\Gamma\,(\Gamma \gg 1)$ the function $g(x)$ is only known numerically. Tables of the function $G(x)$ are given in (C69) for different values of Z and Γ.

Since in a strong field σ is modified by the field itself, it is necessary to know σ in order to estimate τ. Canuto's conductivities have a limited range of validities. When the proton gas becomes degenerate the collision term of the Boltzmann equation has to be treated in a different way to include degeneracy. The modification to Canuto's conductivities due to this phenomenon has been studied recently by Baym *et al.* (BPP69), for the nonrelativistic case and by Kelly (Ke70) and Gentile *et al.* (CG70) for the relativistic case. The general effect is an increase of an amount of the order of $\sim 10^6$ over the previous conductivities ($\approx 10^{23}$ sec^{-1}) for the interior of neutron stars. This increases the magnetic field decay time by the same amount and therefore the pulsar lifetime with respect to the one computed by Gunn and Ostriker (GO69). A different scattering mechanism, i.e., phonon scattering has been recently shown to give the same results as the impurity mechanism used by Canuto, at least for values of $\Gamma \lesssim 75$. A simple analytic expression for σ_0, Equation (10.2) which also reproduces the phonon mechanism, is simply

$$\begin{aligned} \sigma_0 &= 1.08\, T_6^{-1} \varrho_6^{1/3}\, 10^{22} \quad (\mathrm{sec}^{-1}) \\ T_6 &= 10^{-6}\, T\,, \qquad \varrho_6 = 10^{-6}\, \varrho/\mu\,. \end{aligned} \tag{10.5}$$

As discussed earlier, this formula is valid for the neutron star crust (CS70).

A. LONGITUDINAL CASE (ALONG THE MAGNETIC FIELD) (CC69)

In this case the electric field is parallel to the magnetic field. The electric field induces a motion of the electron which will be in the direction of the field. Since the motion of the electron along the field is a free motion, this problem can be treated using the correct wave functions for the electron in a magnetic field. In other words, the Hamiltonian of an electron in an electric field parallel to the magnetic is diagonal in the representation where only a pure magnetic field exists.

The most important scattering process is the Coulomb scattering process

$$e^-_{np_z} + (Z, A) \rightarrow e^-_{n'p_z'} + (Z, A)\,. \tag{10.6}$$

The nucleus can be assumed to be infinitely heavy. In this process n, p_z, and n', p_z' are related by the energy conservation relation,

$$2nH/H_q + (p_z/mc)^2 = 2n'H/H_q + (p_z'/mc)^2\,. \tag{10.7}$$

In the absence of an electric field the motion of the electron is linear in either direction with a zero average motion. When an electric field is applied in the z-axis, the z-motion of the electron will become accelerated in the direction of the field, but this acceleration is dissipated by scattering processes into a constant macroscopic drift velocity v_d. We are therefore interested in the rate that z-momentum of the electron is lost by collision.

For a given set of n and n' (10.7) gives two solutions for p_z:

$$\begin{aligned} &p_z > 0 \qquad p_z' > 0 \\ &p_z > 0 \qquad p_z' < 0\,. \end{aligned}$$

There are two additional sets of solutions with the reverse sign for p_z. These 2 solutions will be included in (10.6) when p_z is integrated from 0 to ∞. The time scale over which the z-momentum is lost is (AA56, AH59, A58):

$$\tau_n^{-1} = \sum_f \left(1 - p_z'/p_z\right) w\,(i \rightarrow f) \tag{10.8}$$

where i is the initial state and f is the final state. This gives:

$$\begin{aligned} \tau_0/\tau_n = &\sum_{n'=0}^{\infty} E\,[E^2 - a_{n'}^2]^{-1/2} \left[1 + \sqrt{\frac{E^2 - a_{n'}^2}{E^2 - a_n^2}}\right] R_+ \\ &+ \sum_{n'=0}^{\infty} E\,[E^2 - a_{n'}^2]^{-1/2} \left[1 - \sqrt{\frac{E^2 - a_{n'}^2}{E^2 - a_n^2}}\right] R_- \end{aligned} \tag{10.9}$$

$$R_\pm \equiv R\,(u, \mp u') \quad u' = (2\theta)^{-1/2}\,(E^2 - a_n^2)^{1/2} \quad a_n^2 = 1 + 2n\theta$$

$$
\begin{aligned}
R(u, u') &= \int_0^\infty \mathrm{d}t \, \frac{|A_{n,n'}(t, u, u')|^2}{[t + (u - u')^2]^2} \, \Phi(\xi) \\
|A_{n,n'}|^2 &= [\omega_1 \Psi(n \mid n') - \omega_2 \Psi(n-1 \mid n'-1)]^2 \\
\Psi(n \mid n') &\equiv (n!\, n'!)^{-1/2} \, e^{-t/2} t^{(n+n')/2} \, {}_2F_0\left(-n', -n; -\frac{1}{t}\right) \\
\Phi(\xi) &= 1 + 3 \int_0^\infty (x\xi)^{-1} \sin(x\xi) \, \mathrm{d}x \\
\xi_\pm &= 2.69 \, \varrho^{-1/3} z^{1/3} \theta^{1/2} [t + \{u \mp u'\}^2]^{1/2} \\
2\omega_1^2 &= 2\omega_2^2 = 1 + E^{-2} \pm E^{-2} (E^2 - a_n^2)^{1/2} (E^2 - a_{n'}^2)^{1/2} ; \\
&\qquad \omega_1 \omega_2 = \theta E^{-2} (nn')^{1/2} .
\end{aligned}
\tag{10.9}
$$

As explained in a review article by Kahn and Frederiske (KF59), the Boltzmann equation (which is the master equation for the computation of conductivity in the field free case) for the longitudinal case is the same as that in the classical case because the electron motion in the z-direction is still free (see, however, KMH65). If f_n is the occupational number for the state n, then the first order solution for the Boltzmann equation is:

$$f_n = f_n^{(0)} + eE\tau_n \partial f_n^{(0)} / \partial p_z$$

where $f_n^{(0)}$ is the equilibrium occupational number for the nth state:

$$f_n^{(0)} = \left\{1 + \exp\left[\frac{\varepsilon_n(x) - \mu}{kT/mc^2}\right]\right\}^{-1} . \tag{10.10}$$

By definition the current J is in the z direction. Since the current associated with a single electron is:

$$j_z = eV_z = ecE^{-1} (E^2 - a_n^2)^{1/2} \tag{10.11}$$

we find

$$J \equiv J_z = -\frac{1}{\hbar} \sum_n \int j_z f_n N(p_z) \, \mathrm{d}p_z \tag{10.12}$$

where $N(p_z)\,\mathrm{d}p_z$ is the density of states between $p_z + \mathrm{d}p_z$ and is given by:

$$N(p_z) \, \mathrm{d}p_z = g (2\pi)^{-2} \lambda_c^{-2} (H/H_q) \, \mathrm{d}p_z . \tag{10.13}$$

The electrical conductivity $\sigma_\parallel$ is obtained from Equations (10.9) and (10.12) and the Ohm's law $J = \sigma_\parallel E$; we obtain

$$\sigma_\parallel = -\hbar^{-1} eg \sum_n \int_{-\infty}^{+\infty} j_z \tau_n (\partial f_n^{(0)} / \partial p_z) \, N(p_z) \, \mathrm{d}p_z . \tag{10.14}$$

Substituting into (10.14) the expression of τ_n (10.8), we find that

$$\sigma_{\parallel} = \bar{\sigma}_{\parallel} (H/H_q)^2 f(\phi, \mu, \theta) \tag{10.15}$$

where

$$\begin{aligned} \bar{\sigma}_{\parallel} &= 4\left[(2\pi)^3 \lambda_c^3 Z \mathcal{N}_i\right]^{-1} (\alpha Z\hbar/mc^2)^{-1} \\ \bar{\sigma} &= 3\bar{\sigma}_0, \quad \text{(Equation 10.2)} \\ \phi &\equiv kT/mc^2, \qquad \theta \equiv H/H_q \end{aligned} \tag{10.16}$$

and

$$\begin{aligned} f(\phi, \mu, \theta) &= \int_1^\infty \frac{\mathrm{d}E}{E^2} \\ &\times \sum_{n=0}^{\infty} \frac{(E^2 - a_n^2)^{1/2} (\partial f_0^{(n)}/\partial E)}{\sum\limits_{n'=0}^{\infty} (E^2 - a_{n'}^2)^{1/2} [1 + g(E)] R_- + \sum\limits_{n'=n}^{\infty} (E^2 - a_{n'}^2)^{1/2} [1 + g(E)] R_+} \\ g^2(E) &= (E^2 - a_{n'}^2)/(E^2 - a_n^2), \quad a_n^2 = 1 + 2n\theta \end{aligned} \tag{10.17}$$

where R_+ and R_- have been defined earlier, Equation (10.9).

The thermal conductivity coefficient λ_H is defined through the relation (M41, M50)

$$Q = -\lambda_H \frac{\mathrm{d}T}{\mathrm{d}x} \tag{10.18}$$

where Q has the dimensions of erg cm^{-2} sec^{-1} and λ_H of erg cm^{-1} deg^{-1}. The Wiedemann Franz law relates the parameter λ_H with the electrical conductivity σ via the relations

$$\lambda_H = \frac{\pi^2}{3} \frac{k^2}{e^2} \sigma T. \tag{10.19}$$

As mentioned earlier, Equation (10.19) has been shown to be valid in a strong magnetic field by Zyrianov (Z64). The conductivity opacity coefficient K_c^H defined as (Ch68):

$$K_c^H = \left(\frac{4ac}{3\varrho}\right) \lambda_H^{-1} T^3 \tag{10.20}$$

becomes:

$$\begin{aligned} K_c^H &= \bar{K}_c^H (H/H_q)^2 f(\phi, \mu, \theta) \\ \bar{K}_c^H &= \tfrac{3}{5} (2\pi)^3 m_H^{-1} \alpha^2 \phi^2 \lambda_c^2 Z^2 \mu_i^{-1} \\ &= 20.26 \times 10^{-8} \, T_6^2 Z^2/A \quad (\mathrm{cm}^2\ \mathrm{g}^{-1}). \end{aligned} \tag{10.21}$$

Equation (10.17) contains a summation in the denominator and even if an approximate expression exists, this approximate expression will be very hard to use. Equation (10.17) has been solved numerically for degenerate and non-degenerate case for $H/H_q \equiv \theta = 1$, $0.1 - 1.2 \leqslant \log \phi \leqslant 0.2$, where $\phi = kT/mc^2$ and $1 \leqslant \mu \leqslant 15$. Extensive numerical tables are given in (CC69). In Figs. 14, 15, 16 we report the behavior of $f(\phi, \mu, \theta)$ for $\phi=0$ and $\theta=1$, 0.1 and for $\phi=1$, $\theta=1$. The discontinuities are due to the density of final states. They almost disappear at high temperature. In Table IV we compare $\sigma_{\parallel}$ with σ_0, Equations (10.2) and (10.15) at $\theta=1$ for densities ranging

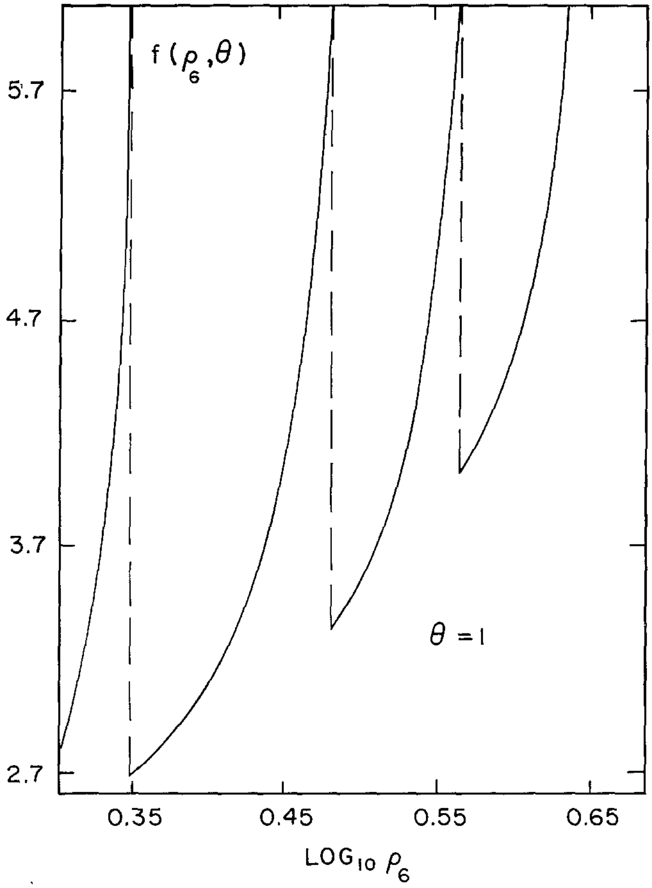

Fig. 14. The function $f(\mu, \theta)$, see Equation (10.15), for a degenerate electron gas as a function of the density ϱ_6 for $\theta \equiv H/H_q = 1$. The relation between ϱ_6 and μ is taken to be $\varrho_6^{2/3} = \pi^2 - 1$. The undulating behavior is related to the density of final states (Figure 3).

TABLE IV

ϱ_6	$\sigma_{\parallel} \times 10^{-21}$ (sec^{-1})	$\sigma_0 \times 10^{-21}$ (sec^{-1})	$\sigma_{\perp} \times 10^{-21}$ (sec^{-1})
8.0	69.37	3.14	1.68
14.7	74.08	3.12	1.13
22.6	71.93	3.32	1.00
31.6	72.61	3.49	0.97
41.6	76.48	3.59	0.98
52.4	81.56	3.45	1.01

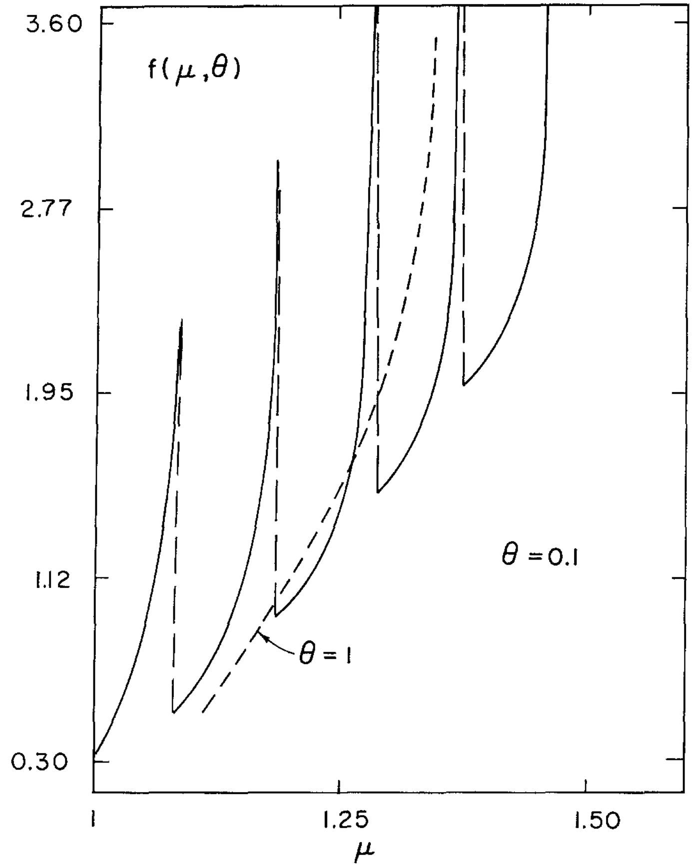

Fig. 15. The same as in Figure 14 for $\theta = 0.1$. In this case the relation between ϱ and μ is more complicated than in the case of Figure 14. The number of discontinuities is noticeable increased with respect to the case $H = H_q$.

between $0.4 \leqslant \log p_6 \leqslant 1.6$. In general it is seen that $\sigma_{\parallel} \gg \sigma_0$. This phenomenon is known as negative magnetoresistence. At higher densities $\sigma_{\parallel}/\sigma_0$ tend to decrease and eventually the density effect will dominate even at $H \simeq H_q$. We therefore expect $\sigma_{\parallel}/\sigma_0 \to 1$ at $\varrho_6 \gg 10^6$ g/cc.

It is seen that the conductivity is increased by the presence of the magnetic field by a factor of 70 at $\theta = 1$. As the density is increased, this difference gradually diminishes and at very high density the effect of the field becomes small and the theory with no magnetic field can be applied.

B. TRANSVERSE CASE ($E_{\perp}$ TO THE MAGNETIC FIELD)

In this direction the motion of the electron is quantized and the ordinary Boltzmann formulation fails. Instead, the Wiedemann-Franz law may be applied to obtain the thermal conductivity from the electrical conductivity. The method of computing

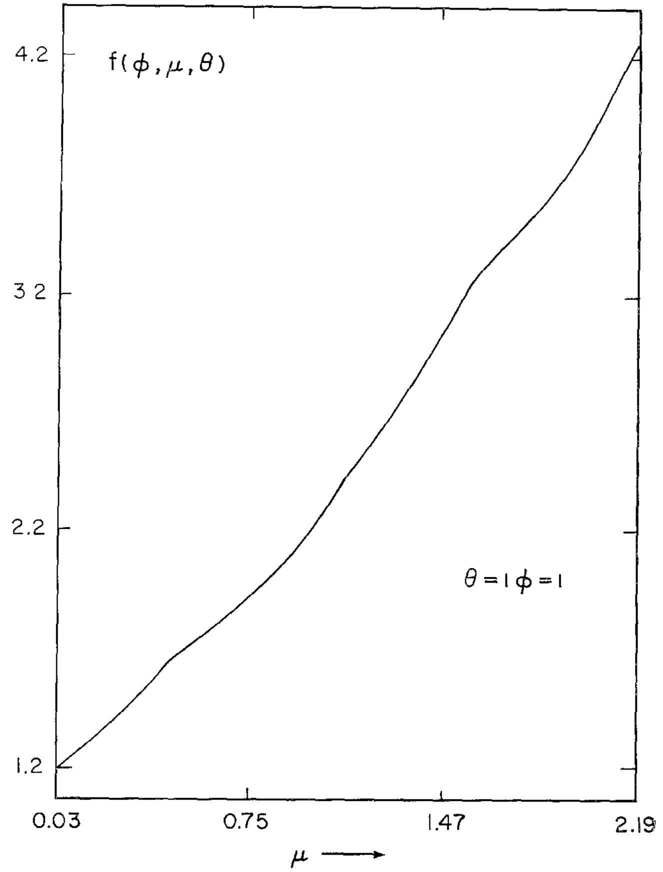

Fig. 16. The funltion $f(\phi, \mu, \theta)$ for $\phi = 1$, $\theta = 1$ as a function of μ. At high temperature the final state discontinuities level off in a very conspicuous manner.

electrical conductivity appropriate to this problem is the density matrix or the 'Kubo formalism'. The general theory has been developed for use in solid state physics for a long time [see KF59].

The average velocities of v_x and v_y are zero for an electron in a magnetic field. An impressed electric field $\perp$ to H will cause particles to move. The non-relativistic solution for an electron in a crossed electric field and magnetic field has been known for some time. The energy eigenvalues and the wave function are given by ($\hbar\omega_0 = = mc^2 H/H_q$) (CCH69, KF59)

$$E_{np_zp_y} = mc^2 (n + \tfrac{1}{2}) \frac{H}{H_q} + \frac{1}{2m} p_z^2 - \frac{eEp_y}{m\omega_0} - \frac{1}{2} \frac{e^2E^2}{m\omega_0^2} \tag{10.23}$$

$$\Psi_{np_zp_y} = \frac{1}{L} \phi_n \left(x + \frac{p_y}{m\omega_0} + \frac{eE}{m\omega_0^2} \right) e^{ik_zz} e^{ik_yy} . \tag{10.24}$$

The wave function ϕ differs from those in the absence of a magnetic field by a displacement from the center by the following given amount:

$$\frac{eE}{m\omega_0^2} = \left(\frac{E}{H}\right)^2 \frac{\hbar}{mc} \tag{10.25}$$

but otherwise have the same analytic form, i.e.,

$$\phi_n(x) = e^{-x^2} H_n(x).$$

The effect of an impressed electric field in the x-direction is therefore to remove the degeneracy with respect to motion in the y-direction, and to shift the center of the orbit.

The conductivity in the $\perp$ direction is no longer a scalar as in the $\parallel$ case. We can define the conductivity tensor as (Z65)

$$\sigma_{\alpha\beta} = \begin{pmatrix} H^{-2} A_{xx} & -H^{-1} A_{yx} & -H^{-1} A_{zx} \\ H^{-1} A_{yx} & H^{-2} A_{yy} & -H^{-1} A_{zy} \\ H^{-1} A_{zx} & H^{-1} A_{zy} & A_{zz} \end{pmatrix} \tag{10.26}$$

where the coefficient A is independent on the magnetic field. The transverse resistance is defined as

$$\varrho_t = \frac{1}{\sigma_{xx}} = \frac{1}{\Delta H^2}\left(A_{yy}A_{zz} + A_{zy}^2\right)$$

where Δ is the determinant of the matrix $\sigma_{\alpha\beta}$. At the lowest power in $1/H$

$$\frac{1}{\sigma_{xx}} \to \frac{A_{yy}A_{zz} + A_{zy}^2}{A_{zz}A_{yx}^2}$$

i.e., the transverse magnetoresistance tends to a constant at high field.

In Figure 17 we show the experimental behavior of $(\Delta\varrho/\varrho_0)_{\parallel,\perp}$ in the case of mercury telluride at 4.2 K as a function of H (G67).

A complete quantum-mechanical computation of the transverse conductivity was performed by using the density matrix approach (CC70). The expression for σ_{yy} and σ_{xy} are given by

$$\sigma_{yy} = \frac{4Z^2\alpha^3}{\sqrt{2\pi}} \frac{mc^2}{\hbar} \lambda_c^3 \mathcal{N}_i \left(\frac{H}{H_q}\right)^{-1/2} f(\mu, \theta) \tag{10.27}$$

$$\sigma_{xy} = \alpha \frac{mc^2}{\hbar} \lambda_c^3 \mathcal{N}_e \left(\frac{H}{H_q}\right)^{-1} \tag{10.28}$$

with

$$f(\mu, \theta) = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \left(1 - \tfrac{1}{2}\delta_{n0}\right)\left(1 - \tfrac{1}{2}\delta_{n'0}\right) F(n, n') \times \left[\int_0^\infty \mathrm{d}s\, \Lambda_- G(s, n, n') + \int_0^\infty \mathrm{d}s\, \Lambda_+ G(s, n, n')\right] \tag{10.29}$$

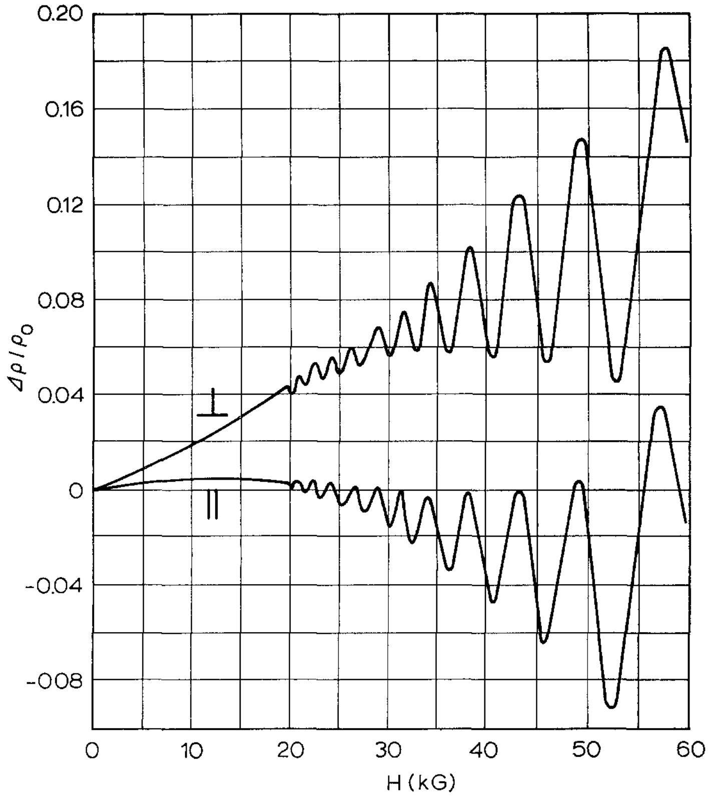

Fig. 17. SdH oscillations in mercury telluride in || and ⊥ magnetic field, at 4.2K; the electron density is $1.18 \times 10t^8/cm^3$. The oscillations are in phase.

$$F(n, n') = \frac{\mu^2}{(\mu^2 - a_n^2)^{1/2} (\mu^2 - a_{n'}^2)^{1/2}} \quad a_n^2 = 1 + 2nH/H_q \equiv 1 + 2n\theta \tag{10.30}$$

$$G_\pm = \frac{s^{1/2}}{\left\{s + \frac{1}{2\theta}[\sqrt{\mu^2 - a_n^2} \pm \sqrt{\mu^2 - a_{n'}^2}]^2\right\}^2}. \tag{10.31}$$

The function $\Lambda_\pm$ depends on a certain combination of spinor coefficients as reported in the appendix of the original paper (CC70). In Figure 18 the function $f(\mu, \theta)$ is given *vs.* $\mu = (1 + \varrho_6^{2/3})^{1/2}$ for $H = 10^{-1} H_q$. Compared with the equivalent function for the longitudinal case (Figure 14), we see that in each jump the behaviour is just the opposite. The $\theta = 1$ case is perfectly analogous although the number of jumps per density interval is reduced. Once we know the function $f(\mu, \theta)$ the transverse conductivity can be easily computed from

$$\sigma_\perp = \frac{\sigma_{xy}^2 + \sigma_{yy}^2}{\sigma_{yy}}. \tag{10.32}$$

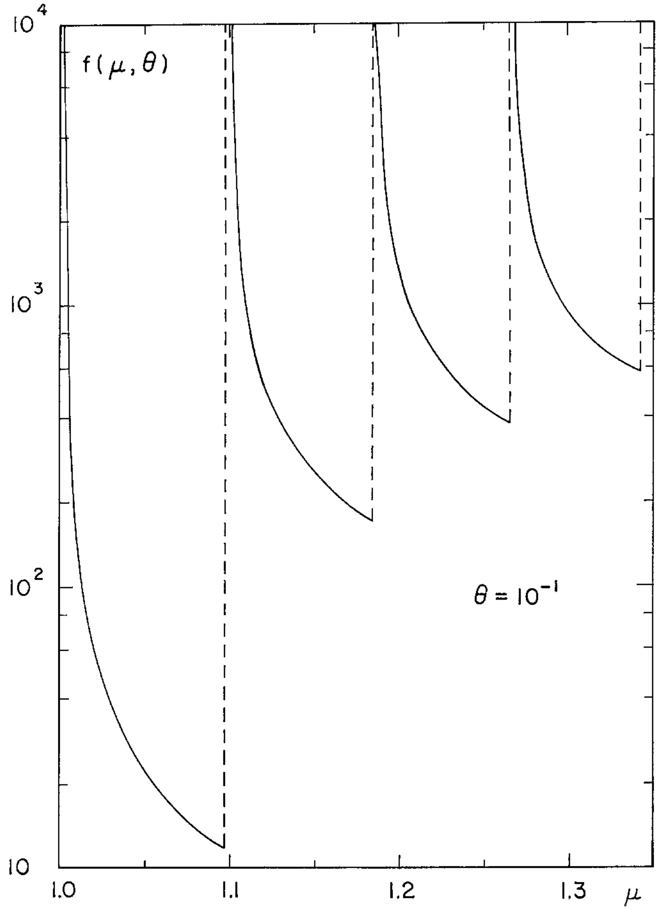

Fig. 18. The function $f(\mu, \theta)$ vs μ (in units of mc^2) for $\theta = H/H_q = 10^{-1}$.

A comparison of the transverse, zero-field and longitudinal conductivities, Equations (10.32), (10.2) and (10.15) is given in Table IV, *vs.* density at $H = H_q$. A sizable effect can be found only if the density is low enough. At higher density the three conductivities will coincide and no magnetic effect would be left whatsoever. This situation is analogous to the one encountered in the neutron beta decay (see Figure 13).

11. Magnetization of an Electron Gas. Semi-Permanent Magnetism (LOFER)

An electron possesses a magnetic moment $\mu_\beta = e\hbar/2mc$ (Bohr magneton) and macroscopic magnetization can result from this magnetic moment and from the orbital motion of the electrons. As is well known in plasma physics, the magnetic moment associated with a classical electron is diamagnetic. However, under certain circumstances an electron gas can possess a net magnetic moment.

In previous sections we have made no distinctions between B, the magnetic induction, and the external field H. The properties of an electron depend only on the strength of field near the electron and this is the field we talk about. However, as has been shown previously, (Ra44, Wa47), an orbiting electron in a magnetic field senses not the external field H but the magnetic induction B such that

$$B = H + 4\pi M \tag{11.1}$$

where M is the magnetic moment of the gas. As a result the Larmor frequency of the electron ω_L is given by:

$$\omega_L = \frac{eB}{mc} \quad \text{and not} \quad \frac{eH}{mc}. \tag{11.2}$$

If there is a net magnetic moment of an electron gas, M, then $B \neq H$. However, as M depends on ω_L which in turn depends on B, M must be a function of B. Hence the magnetic moment of an electron gas must be given by the following non-linear equation:

$$B = H + 4\pi M(B). \tag{11.3}$$

Under ordinary conditions (i.e., high temperature and low density) $4\pi M \ll H$, i.e.,

$$\frac{B}{H} = \mu = 1 + 4\pi \frac{M}{H} \simeq 1 \tag{11.4}$$

and no distinction need be made between B and H.

At high density and low temperature, however, as we will show below, the value of M increases and eventually $4\pi M(B)$ becomes greater than B and solutions can exist such that

$$B = 4\pi M(B), \qquad H = 0. \tag{11.5}$$

This means that an electron gas can become self-magnetized. Such a state of permanent magnetization is distinctively different from the ordinary ferromagnetism. The magnetization associated with Equation (11.5) will be referred to as 'Landau Orbital Ferromagnetism' or LOFER (LCCC69, CCCL69).

The magnetic moment M of an electron gas has been shown to be (CC68c):

$$M = \frac{1}{\Omega} \frac{1}{kT} \frac{\partial}{\partial H} \ln Z(T, H) \tag{11.6}$$

where Ω is the volume of the system and Z is the grand partition function, which is given by

$$\frac{1}{\Omega} \ln Z = \frac{eB}{\pi\hbar} \left\{ \frac{1}{2} \int_{-\infty}^{+\infty} dx \ln\left[1 + \exp\left\{-\frac{\varepsilon(x, 0) - \mu}{T/T_0}\right\}\right.\right.$$
$$\left.\left. + \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} dx \ln\left[1 + \exp\left\{-\frac{\varepsilon(x_1 n) - \mu}{T/T_0}\right\}\right]\right\} \tag{11.7}$$

where
$$\varepsilon^2(x, n) = 1 + x^2 + 2nB/B_q \quad x \equiv p_z/mc, \quad B_q \equiv H_q. \tag{11.8}$$

After substituting (11.7) into (11.6) and carrying out some reductions as in reference (CC68c) we find:

$$M \equiv \frac{2}{\pi^2}\frac{\mu_B}{\lambda_c^3}\left\{\frac{1}{2}C_2(T,\mu) + \sum_{n=1}^{\infty} a_n^2 C_2(T/a_n, \mu/a_n) - \frac{B}{B_q}\sum_{n=1}^{\infty} n\, C_1(T/a_n, \mu/a_n)\right\} \tag{11.9}$$

or

$$M = \frac{B_q}{B}\frac{\mu_B}{mc^2}(P_\parallel - P_\perp)$$

where $P_\parallel$ is the parallel stress and $P_\perp$ is the normal stress. C_n are the functions defined earlier (see Equations 5.27–5.30).

A. CLASSICAL LIMIT

We will show that at high temperature and low density M does reduce to the Curie-Langevin law for the magnetic susceptibility. In the non-degenerate and non-relativistic case:

$$\varepsilon(x, n) \to 1 + \tfrac{1}{2}x^2 + nB/B_q \quad \exp\left(\frac{\varepsilon - \mu}{T/T_0}\right) \gg 1. \tag{11.10}$$

In the definition of the C_k function when 1 is neglected against $\exp[(\varepsilon - \mu)/(T/T_0)]$, Equation (11.9) becomes:

$$\left[M = \frac{2}{\pi^2}\frac{\mu_B}{\lambda_c^3}\frac{1}{2}\left(\frac{T}{T_0}\right)\exp\frac{\mu - 1}{T/T_0} + \frac{T}{T_0}\sum_{n=1}^{\infty}\exp\left(\frac{\mu - 1}{T/T_0} - \frac{nB/B_q}{T/T_0}\right) - \frac{B}{B_q}\sum_{n=1}^{\infty} n\exp\left(\frac{\mu - 1}{T/T_0} - \frac{nB/B_q}{T/T_0}\right)\right]\int_0^{\infty}\exp\left(-\frac{x^2}{T/T_0}\right)\mathrm{d}x. \tag{11.11}$$

Equation (11.11) can be rearranged into a geometrical series and its derivatives. After carrying out the sum we find:

$$M = \frac{(2\pi T/T_0)^{1/2}\,\mu_B}{\pi^2\lambda_c^3}\exp\frac{\mu - 1}{T/T_0}\frac{\partial}{\partial\eta}(\eta\coth\eta) \tag{11.12}$$

$$\eta \equiv \frac{B\mu_B}{kT}. \tag{11.13}$$

In the case $\eta \ll 1$ we then find:

$$\frac{\partial}{\partial\eta}(\eta\coth\eta) \to \tfrac{2}{3}\eta. \tag{11.14}$$

As we will show later $B \sim H$ and $M \ll B$ so that in (11.12) B may be replaced by H. We therefore find for the magnetic susceptibility

$$\chi \equiv \left(\frac{M}{H}\right)_{H=0} = \frac{2}{3} \frac{\mu_B^2}{kT} \exp \frac{\mu - 1}{T/T_0} \left[\frac{1}{4\pi^3} \left(\frac{2\pi kT}{mc^2}\right)^{3/2} \frac{1}{\lambda_c^3}\right]. \tag{11.15}$$

From the definition of Z we find that in the limit (11.10)

$$\ln Z = \frac{\Omega}{4\pi^3} \frac{1}{\lambda_c^3} \left(\frac{2\pi kT}{mc^2}\right)^{3/2}. \tag{11.16}$$

The particle density N is given by

$$N = \frac{1}{\Omega} \lambda \frac{\partial}{\partial \lambda} \ln Z, \qquad \lambda \equiv \exp \frac{\mu - 1}{T/T_0} \tag{11.17}$$

and therefore

$$\chi = \tfrac{2}{3} \left(\frac{\mu_B^2}{kT}\right) N \propto \frac{1}{T} \tag{11.18}$$

which is the Curie-Langevin law.

This classical result includes both the diamagnetic part (due to particle orbital motions) and the paramagnetic part (due to spin magnetic moment). The paramagnetic part is $N \mu_B/kT$ and the diamagnetic part is minus $\frac{1}{3}$ that of the paramagnetic part, and the sum of the two yields a factor of $\frac{2}{3}$ in (11.18).

It is difficult to separate the general expression (11.9) into the corresponding diamagnetic and paramagnetic parts. Nevertheless we can still decompose (Gordon decomposition) the 4 current j_μ (Sa67)

$$j_\mu = ie\bar{\psi}\gamma_\mu\psi = j_\mu^{(1)} + j_\mu^{(2)} \tag{11.19}$$

such that

$$j_\mu^{(1)} A_\mu = \frac{ie\hbar}{2m} [\partial_\mu \bar{\psi}\psi - \psi \partial_\mu \bar{\psi}] A_\mu - \frac{e^2}{mc} A_\mu^2 \bar{\psi}\psi \tag{11.20}$$

becomes the Landau Hamiltonian giving rise to diamagnetism in the non-relativistic limit and the other part

$$\begin{aligned} j_\mu^{(2)} A_\mu &= - \frac{e\hbar}{2mc} \left[\frac{1}{2} \frac{\partial A_\mu}{\partial x_\nu} (\bar{\psi}\sigma_{\mu\nu}\psi) + \frac{1}{2} \frac{\partial A_\nu}{\partial x_\mu} (\bar{\psi}\sigma_{\mu\nu}\psi)\right] \\ \sigma_{\mu\nu} &= - i\gamma_\mu\gamma_\nu \quad \mu \neq \nu \end{aligned} \tag{11.21}$$

becomes the Pauli Hamiltonian giving rise to spin paramagnetism in the non-relativistic limit.

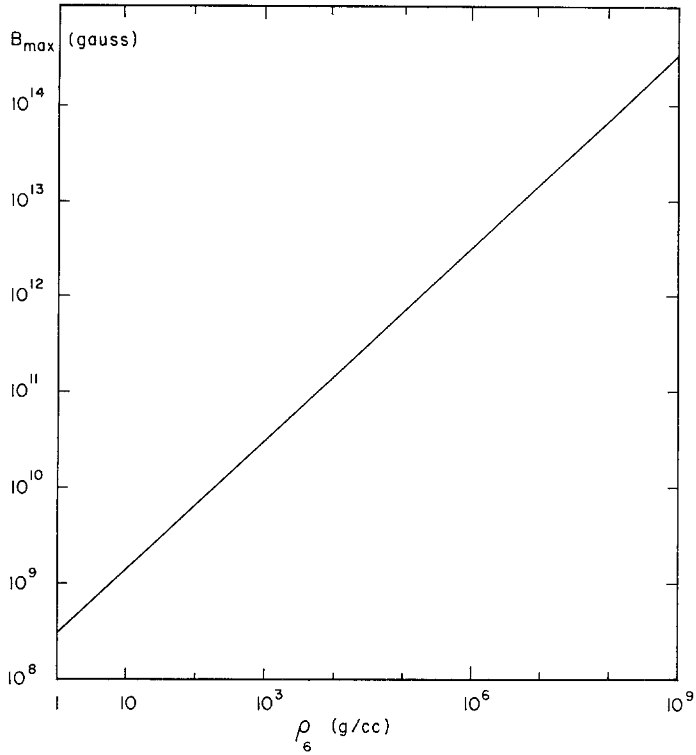

Fig. 19. As explained in the text at a fixed density (or μ) Equation (11.22) admits many solutions because of the oscillatory character of $M(B)$. In this figure only the maximum values of B are plotted vs the corresponding densities. It can be checked that $B_{\max} \simeq \varrho_6^{2/3}$ which is what the flux conservation law would predict.

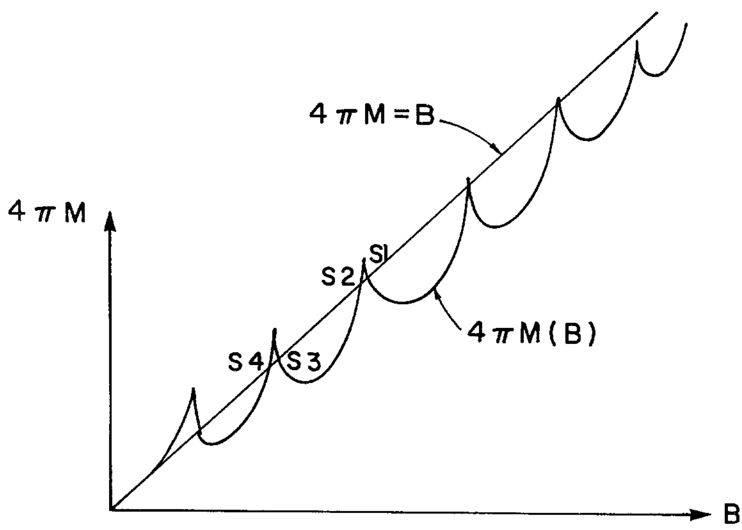

Fig. 20. Behaviour of the two functions $4\pi M$ and $4\pi M(B)$ as given by Equation (11.22). The last point of intersection corresponding to B_{ax} is plotted in Figure 19.

B. DEGENERATIVE EXPRESSIONS FOR M

This limit is analogous to that in the equation of state. We find

$$\begin{aligned} M &= \frac{2}{\pi^2}\frac{\mu_B}{\lambda_c^3}\left[\frac{1}{2}C_2(\mu) + \sum_{n=1}^{s} a_n^2 C_2(\mu/a_n) - \frac{B}{B_q}\sum_{n=1}^{s} nC_1(\mu/a_n)\right] \\ &= \frac{B}{B_q}\frac{\mu_B}{mc^2}\left[P_\parallel(B) - P_\perp(B)\right]. \end{aligned} \tag{11.22}$$

$P_\parallel$ and $P_\perp$ show oscillatory behavior as a function of the density N, and so will M. This oscillatory behavior is entirely due to orbital quantization and is responsible for the existence of a state of permanent magnetism, as we shall now show.

C. LOFER (LANDAU ORBITAL FERROMAGNETISM)

This is a quasi-stable self-consistent macroscopic magnetism associated with a non-interacting degenerate electron gas. (This state is characterized by an energy higher than the ground state.) This macroscopic magnetization is the sum of all microscopic magnetizations associated with electrons in their respective Landau levels while the Landau levels of the system are in turn maintained by the macroscopic magnetization of the system. The magnitude of the field is given by the solution of Equation (11.3) with $H=0$, i.e.,

$$B = 4\pi M(B) \quad \text{or} \quad M = M(4\pi M). \tag{11.23}$$

$M(B)$ or $M(4\pi M)$ is derived by using in the interaction terms of the Hamiltonian-**J**·**A**, the induced current **J** and the vector potential A due to the orbital motions of all electrons. There are a number of solutions of the LOFER state for a given density of electrons and the maximum value of LOFER magnetism is shown in Figure 14 for densities ranging from 10^6 g/cm^3 to 10^{15} g/cm^3.

Thermodynamically the LOFER state is not the minimum energy state and hence not the most stable state in the thermodynamical sense. But a system that is not thermodynamically stable may still require a very long time to reach the most stable state. (An example is a bottle of hydrogen gas at room temperature; the thermodynamically most stable state is when all hydrogen nuclei are catalyzed into a piece of iron, but the lifetime against such a transition is over 10^{100} years, and one can regard a bottle of hydrogen at room temperature as a truly stable state for practically all purposes.)

The LOFER state is the solution of the equation $B=4\pi M(B)$ and in the $4\pi M$ *vs* B plot, the LOFER states are the intersections of the curves $B=4\pi M$ and $B=4\pi M(B)$. Since $4\pi M(B)$ is a spiked function, there are two solutions S_1 and S_2 associated with each spike as shown in Figure 15. As will be discussed later, a state with a higher magnetization is the more stable one. If the systems attains the solution S_2 then it is certainly unstable against a transition into the next lower level S_3. The stability of this system is therefore the same as the stability against a transition from the S_1 state into the S_2 state (or S_3 into S_4, and so on).

In general $M(B)$ is also a function of the temperature and at high temperature $M(B)$ reduces to the classical expression, for which no intersection exists between the curve $4\pi M(B)$ and $4\pi M$, therefore there must exist a critical temperature T_c above which no LOFER state exists. This is analogous to the classical case of Ferromagnetism in which above the Curie temperature only paramagnetism exists. The value of the critical temperature has not been ascertained for the non-relativistic case. It is found that the condition for LOFER to exist is:

$$\mu/\kappa T > 10^7 (c/v_F)^2 \tag{11.24}$$

where μ is the Fermi energy and v_F the Fermi velocity. For white dwarfs (for which (11.24) is barely applicable) $v_F \simeq c$. The condition for LOFER magnetism to exist is therefore $T \lesssim 10^3$ K ($\mu \simeq mc^2$ in the center of most white dwarfs). However, (11.24) expresses a much more stringent conditions for the existence of LOFER state. The actual transition temperature is probably $\simeq 10^4$ K. This temperature is achievable in white dwarfs after a few times 10^9 years if crystallization does take place (AM59, S61, MR67).

Below this temperature the LOFER state is stable. In order for the gas to make a transition form S_1 to S_2 it is necessary to cross an energy barrier of the amount $\beta \Delta\Omega_B \cdot V$ where V is the characteristic volume of the system, and $\Delta\Omega_B$ is the free energy difference at the peak of the $M(B)$ curve and at S_1. If we choose $V = 4\pi R_L^3/3$ then it is shown that below T_c

$$V\Delta\Omega_B \gg \kappa T$$

so that the probability for the system to make a transition from S_1 to S_2 which is proportional to $\exp(-V\Delta\Omega_B/\kappa T)$, is negligibly small.

12. Coulomb Bremsstrahlung in a Magnetized Plasma

In addition to the synchrotron radiation process (Section 6) the bremsstrahlung process is an important radiation process. Further, it is the most important continuum emission process.

Unlike the synchrotron radiation the bremsstrahlung process must take place in the presence of a plasma. The presence of a plasma strongly affects the propagation of electromagnetic radiation. In this section the bremsstrahlung process in the presence of a plasma and a magnetic field will be discussed in detail.

Consider a test particle moving in a magnetized plasma. All possible effects can be understood in terms of the equation of motion whose solution gives us the current $j_\alpha(r, t)$ ($\alpha = 1, 2, 3$). Quantum mechanically this current is given by

$$j_\alpha(r, t) = \frac{ie\hbar}{2m} [\Psi^* \tilde{\nabla}_\alpha \Psi - \Psi \tilde{\nabla}_\alpha \Psi^*] \qquad \tilde{\nabla} \equiv \nabla - \frac{ie}{\hbar c} A_{\text{ext}} \tag{12.1}$$

where Ψ describes the state of the particle. If the particle is thought to be acted upon only by an external magnetic field the Ψ to be used is the solution of Schrodinger or Dirac equation with a magnetic field [see Appendix I]. This particle, moving in a medium produces an electromagnetic field, which in turn can be determined, using the Maxwell equation, as a function of $j_\alpha(r, t)$ itself. In a magnetic medium the electric field produced by such a current is given by [Sh66]

$$\begin{aligned} \mathbf{E}(r,t) &= \int \mathbf{E}(\kappa,\omega)\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\, \mathrm{d}^3\kappa\, \mathrm{d}\omega \\ \mathbf{E}(\kappa,\omega) &\equiv \sum_{l=1}^{2} \frac{4\pi i}{\omega} \frac{\mathbf{a}(l)\, \mathbf{j}\cdot\mathbf{a}^*(l)}{\kappa^2 c^2/\omega^2 - n_l^2}. \end{aligned} \tag{12.2}$$

The quantity n_l^2, the refractive index for the ordinary ($l=1$) and extraordinary wave ($l=2$) respectively, is given by

$$n_l^2 = -\frac{B}{2A} \pm \frac{1}{2A}\left(B^2 - 4AC\right)^{1/2} \tag{12.3}$$

$$\begin{aligned} A &= \varepsilon_{11}\sin^2\theta + \varepsilon_{33}\cos^2\theta + \varepsilon_{13}\sin^2\theta \\ B &= (\varepsilon_{12}\sin\theta - \varepsilon_{33}\cos\theta)^2 + \varepsilon_{13}^2(\cos^2 2\theta + \sin^4\theta) \\ &\quad - \varepsilon_{11}\varepsilon_{33} - \varepsilon_{22}\left[\varepsilon_{11}\sin^2\theta + \varepsilon_{33}\cos^2\theta + \varepsilon_{13}\sin 2\theta\right] \\ C &= \varepsilon_{11}\varepsilon_{22}\varepsilon_{33} - \varepsilon_{33}\varepsilon_{12}^2 - \varepsilon_{11}\varepsilon_{23}^2 - \varepsilon_{22}\varepsilon_{13}^2 - 2\varepsilon_{12}\varepsilon_{23}\varepsilon_{13}. \end{aligned}$$

Here θ is the angle between κ and H. The dielectric tensor of the medium, $\varepsilon_{\alpha\beta}$, is usually given in a system (x_0, y_0, z_0) in which H is along the z_0-axis and κ lies in the plane (x_0, z_0).
The vector $\mathbf{a}$ has been shown by Shafranov to be

$$\mathbf{a} = a_y\{i\alpha_x^0, 1, i\alpha_z^0\} \tag{12.5}$$

where $[a_y^2 = (1+\alpha_x^2)^{-1}]$ the quantities α_x^0 and α_z^0 given by

$$\alpha_x^0 = \frac{\varepsilon_{12}\varepsilon_{33} + \varepsilon_{23}\varepsilon_{13} - n_l^2\left[\varepsilon_{12}\sin\theta - \varepsilon_{23}\cos\theta\right]\sin\theta}{n_l^2(\varepsilon_{11}\sin^2\theta + \varepsilon_{33}\cos^2\theta + \varepsilon_{13}\sin 2\theta) - \varepsilon_{11}\varepsilon_{33} + \varepsilon_{13}^2} \tag{12.6}$$

$$\alpha_z^0 = -\frac{\varepsilon_{12}\varepsilon_{13} + \varepsilon_{11}\varepsilon_{23} + n_l^2\left[\varepsilon_{12}\sin\theta - \varepsilon_{23}\cos\theta\right]\cos\theta}{n_l^2(\varepsilon_{11}\sin^2\theta + \varepsilon_{33}\cos^2\theta + \varepsilon_{13}\sin 2\theta) - \varepsilon_{11}\varepsilon_{33} + \varepsilon_{13}^2} \tag{12.7}$$

are the ratio of the component of the electric field and therefore determine the polarization of the wave. In fact $\alpha_x^0 = \alpha_x\cos\theta + \alpha_z\sin\theta$, $\alpha_z^0 = a_z\cos\theta - \alpha_x\sin\theta$ satisfy the following equation

$$\alpha_x^2 + i\,\frac{\eta_{xx} - \eta_{yy}}{\eta_{yx}}\,\alpha_x - 1 = 0 \tag{12.8}$$

with

$$\begin{aligned}\eta_{xx} &= N[\varepsilon_{11}\varepsilon_{33} - \varepsilon_{13}^2]\\ \eta_{yy} &= N\{\varepsilon_{22}[\varepsilon_{11}\sin^2\theta + \varepsilon_{33}\cos^2\theta] - \varepsilon_{12}^2\sin^2\theta\\ &\qquad + [\varepsilon_{22}\varepsilon_{13} + \varepsilon_{23}\varepsilon_{12}]\sin 2\theta - \varepsilon_{23}^2\cos^2\theta\} \qquad (12.9)\\ \eta_{yx} &= -\,iN\{[\varepsilon_{12}\varepsilon_{33} + \varepsilon_{23}\varepsilon_{13}]\cos\theta + [\varepsilon_{12}\varepsilon_{13} + \varepsilon_{23}\varepsilon_{11}]\sin\theta\}\\ N^{-1} &= \varepsilon_{11}\sin^2\theta + \varepsilon_{33}\cos^2\theta + \varepsilon_{13}\sin 2\theta\,.\end{aligned}$$

Depending on the angle θ, Equation (12.8) gives the various types of polarizations. Once the electric field is known as a function of j_α, the Maxwell equation gives the definition of the intensity per second or power emitted by the particle, as

$$I = \frac{1}{T}\int \frac{\mathrm{d}^3\kappa\,\mathrm{d}\omega}{(2\pi)^4}\,[j_\alpha(k,\omega)\,E_\alpha^*(k,\omega) + E_\alpha(k,\omega)\,j_\alpha^*(k,\omega)] \qquad (12.10)$$

Substituting Equation (12.2) into Equation (12.10) we obtain

$$I = \int_0^\infty \mathrm{d}\omega \int_{4\mu} \mathrm{d}\Omega\, I(\omega, \Omega) \qquad (12.11)$$

where the emissivity $I(\omega, \Omega)$ is defined as

$$I(\omega, \Omega) = 2\,\mathrm{Re}\int \mathrm{d}^3\kappa\, G_{\alpha\beta}(k,\omega)\, L_{\alpha\beta}(k,\omega) \qquad (12.12)$$

with

$$\begin{aligned}G_{\alpha\beta} &= j_\alpha(k,\omega)\,j_\beta^*(k,\omega)\\ L_{\alpha\beta} &= \frac{4\pi i}{\omega}\sum_{l=1}^{2}\frac{a_\alpha(l)\,a_\beta^*(l)}{k^2c^2/\omega^2 - n_l^2}\,.\end{aligned} \qquad (12.13)$$

In Shafranov's paper many examples with different $j_\alpha(k, \omega)$ are discussed in detail. To compute Equation (12.11) in the case of Coulomb bremsstrahlung in a magnetic field, i.e.,

$$e^- + (Z, A) \to e^- + (Z, A) + \gamma$$

we must compute the current given in Equation (12.1). From perturbation theory we know that

$$\Psi_\alpha(r) = \psi_\alpha(r) - \sum_\beta \frac{\langle\beta|\, A_0\,|\alpha\rangle}{E_\beta - E_\alpha}\,\psi_\alpha(r) \qquad (12.14)$$

where

$$A_0 = Ze/r\,.$$

The Fourier transform of the electric current, $j(k, \omega)$, is simply given by

$$j(k,\omega) = 2\pi\hbar\, j(k)\,\delta(E_i - E_f - \hbar\omega) \qquad (12.15)$$

with ($\alpha = x, y, z$)

$$j_\alpha(k) = \frac{e}{2m} \sum_I \left\{ \frac{\langle f | \, e^{-ik\cdot r} \Pi_\alpha \, |I\rangle \, \langle I | \, A_0 \, |i\rangle}{E_i - E_I - i\Gamma} + \frac{\langle f | \, A_0 \, |I\rangle \, \langle I | \, e^{-ik\cdot r} \Pi_\alpha \, |i\rangle}{E_i - E_I - \hbar\omega} \right\} \tag{12.16}$$

where i, f and I stand for initial, final and intermediate state.

A. DIELECTRIC TENSOR FOR A COLD PLASMA

With Equation (12.16) substituted in Equation (12.2) and then in Equation (12.10) one obtains in principle the energy loss at any angle and for any form of dielectric tensor. The problem is quite involved and the final form too complicated to analyze. We will therefore study separately the propagation $\parallel$ and $\perp$ to H as usually done in magnetized plasma, and will specify the type of plasma to be worked with. We will use the tensor $\varepsilon_{\alpha\beta}$ as given in the magneto-ionic theory, i.e., (S62)

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix} \tag{12.17}$$

where

$$\begin{aligned} R &= 1 - \frac{\omega_p^2}{\omega^2} \frac{\omega}{\omega - \omega_H} - \frac{\Omega_p^2}{\omega^2} \frac{\omega}{\omega + \Omega_H} \\ L &= 1 - \frac{\omega_p^2}{\omega^2} \frac{\omega}{\omega + \omega_H} - \frac{\Omega_p^2}{\omega^2} \frac{\omega}{\omega - \Omega_H} \\ P &= 1 - \frac{\omega_p^2}{\omega^2} - \frac{\Omega_p^2}{\omega^2} \qquad 2S = R + L \qquad 2D = R - L\,. \end{aligned} \tag{12.18}$$

The various symbols are defined in the following way

$$\begin{aligned} \omega_p^2 &= \frac{4\pi N_e e^2}{m} & \omega_H &= \frac{eH}{mc} \\ \Omega_p^2 &= \omega_p^2 \frac{m}{M_i} Z & \Omega_H &= \omega_H \frac{m}{M_i} Z_i \end{aligned} \tag{12.19}$$

where M_i is the mass of the ion, and Z_i its charge, $n_i Z_i = n_e$. With this notation, Equation (12.3) for H_l^2 reduces to

$$N_l^2 = \frac{1}{2A} (B \pm F) \tag{12.20}$$

with

$$\begin{aligned} A &= S \sin^2\theta + P \cos^2\theta & B &= RL \sin^2\theta + S(1 + \cos^2\theta) \\ C &= PRL & F^2 &= (RL - PS)^2 \sin^4\theta + 4P^2 D^2 \cos^2\theta \end{aligned} \tag{12.21}$$

Equation (12.20) can also be written in a more transparent form (S62)

$$\operatorname{tg}^2\theta = -\frac{P(N_x^2 - R)(N_0^2 - L)}{(SN_x^2 - RL)(N_0^2 - P)} \tag{12.22}$$

from which the dispersion relation at $\theta = 0$ and $\theta = \pi/2$ are easily obtained as

$$\begin{array}{lll} \theta = \pi/2 & N_0^2 = P \text{ (ordinary)} & N_x^2 = RL/S \text{ (extraordinary)} \\ \theta = 0 & N_0^2 = L & N_x^2 = R\,. \end{array} \tag{12.23}$$

The terminology, ordinary and extraordinary, is well established at $\theta = \pi/2$ where it is seen that the minus sign in Equation (12.20) gives rise to $N_-^2 = P$, independent of the magnetic field: i.e., this mode propagates as it would in the absence of H (ordinary). The plus sign gives rise to $N_+^2 = RL/S$, i.e., it does depend on H (extraordinary). The same terminology is retained for $\theta = 0$. The parameters α, α_x and α_z are easily transformed to ($l = \pm$)

$$\begin{aligned} \alpha(l) &= -PD\cos\theta\,[AN_l^2 - PS]^{-1} \\ \alpha_x(l) &= D(N_l^2\sin^2\theta - P)(AN_l^2 - PS)^{-1} \\ \alpha_z(l) &= N_l^2 D\sin\theta\cos\theta\,(AN_l^2 - PS)^{-1}\,. \end{aligned} \tag{12.24}$$

B. PROPAGATION AT $\theta = 0$.

From the previous equations one can easily see that at $\theta = 0$, one obtains

$$\begin{array}{lll} A = P & B = 2PS & C = PRL \\ N_0^2 = L & N_x^2 = R & \alpha(0) = -\alpha(X) = 1 \\ \alpha_z(0) = \alpha_z(X) = 0 & & \alpha_x(0) = -\alpha_x(X) = 1 \end{array} \tag{12.25}$$

i.e.,

$$\begin{aligned} \mathbf{a}(0) &= \frac{1}{\sqrt{2}}[i, 1, 0] \\ \mathbf{a}(x) &= \frac{1}{\sqrt{2}}[-i, 1, 0] \end{aligned}\,. \tag{12.26}$$

From here it follows that since $iE_x/E_y = -\alpha_x$, the ordinary wave is left-hand circularly polarized. This means that at $\theta = 0$, only circularly polarized waves can propagate in a plasma. However, the emerged radiation may still be unpolarized or linearly polarized as a result of a combination of both R- and L-polarization states.

Substituting the polarization vector a into the expression for the electric field a little algebra gives ($\alpha = 1, 2, 3$)

$$E_\alpha j_\alpha^* + E_\alpha^* j_\alpha = \frac{4\pi^2}{\omega}\,\delta(\Lambda)\,|j_x - ij_y|^2 \tag{12.27}$$

for the extraordinary wave and

$$E_\alpha j_\alpha^* + E_\alpha^* j_\alpha = \frac{4\pi^2}{\omega}\,\delta(\Lambda)\,|j_x + i j_y|^2 \tag{12.28}$$

for the ordinary wave. The delta function comes from the fact that

$$\begin{aligned}&\frac{1}{\Lambda} = P(\Lambda) - i\pi\delta(\Lambda)\\ &\Lambda = \kappa^2 c^2/\omega^2 - N_l^2\end{aligned} \tag{12.29}$$

where P stands for the principal value.

Remembering that (ST68, AR69)

$$\begin{aligned}&(\Pi_x - i\Pi_y)\,|n\rangle = mc\,(2nH/H_q)^{1/2}\,|n-1\rangle\\ &(\Pi_x + i\Pi_y)\,|n\rangle = mc\,[2(n+1)\,H/H_q]^{1/2}\,|n+1\rangle\end{aligned} \tag{12.30}$$

we obtain for the extraordinary wave

$$\begin{aligned}\frac{\mathrm{d}I(\omega,\Omega)}{\mathrm{d}\omega\,\mathrm{d}\Omega} &= I_0\,\frac{\omega^2 N}{\sqrt{\varepsilon - \omega - n'\omega_H}}\int_0^\infty \frac{\mathrm{d}t}{(t+\lambda)^2}\\ &\times\left\{\frac{1}{E_1}\sqrt{1+n'}\,\mathscr{I}(n, n'+1, t) + \frac{1}{E_2}\sqrt{n}\,\mathscr{I}(n-1, n')\right\}^2\end{aligned} \tag{12.31}$$

$$\begin{aligned}&I_0 = \frac{Z^2\alpha^3}{\sqrt{2}\,8\pi}\,\frac{mc^2}{\hbar/mc^2}\,N_i\lambda_c^3\\ &N^2 = R, \qquad \varepsilon = E_i/mc^2, \qquad \omega \equiv \omega/(mc^2/\hbar), \qquad \omega_H \equiv H/H_q\\ &\lambda = (2\omega_H)^{-1}\{\sqrt{2(\varepsilon - n\omega_H)} - \omega N - \gamma\sqrt{2(\varepsilon - \omega - n'\omega_H)}\}^2\\ &\gamma = \pm 1\end{aligned}$$

$$\mathscr{I}(n, n', t) = (n!\,n'!)^{-1/2}\,e^{-t/2}\,t^{(n+n')/2}\;{}_2F_0\left(-n, -n'; -\frac{1}{t}\right) \tag{12.32}$$

$$\begin{aligned}&E_1 = \omega - \omega_H - \tfrac{1}{2}\omega^2 N^2 - \gamma\omega N\sqrt{2(\varepsilon - \omega - n'\omega_H)}\\ &E_2 = \omega - \omega_H + \tfrac{1}{2}\omega^2 N^2 - \gamma\omega N\sqrt{2(\varepsilon - n\omega_H)}\,.\end{aligned}$$

A perfectly analogous computation gives the following result for the ordinary wave

$$\begin{aligned}\frac{\mathrm{d}I(\omega,\Omega)}{\mathrm{d}\omega\,\mathrm{d}\Omega} &= I_0\,\frac{\omega^2 N}{\sqrt{\varepsilon - \omega - n'\omega_H}}\int_0^\infty \frac{\mathrm{d}t}{(t+\lambda)^2}\\ &\times\left\{\frac{1}{E_3}\sqrt{n'}\,\mathscr{I}(n'-1, n, t) + \frac{1}{E_4}\sqrt{n+1}\,\mathscr{I}(n+1, n', t)\right\}^2\end{aligned} \tag{12.33}$$

$$\begin{aligned}
E_3 &= \omega + \omega_H - \tfrac{1}{2}\omega^2 N^2 - \gamma\omega N\sqrt{2(\varepsilon - \omega - n'\omega_H)} \\
E_4 &= \omega + \omega_H + \tfrac{1}{2}\omega^2 N^2 - \gamma\omega N\sqrt{2(\varepsilon - n\omega_H)} \\
N^2 &= L\,.
\end{aligned} \tag{12.34}$$

Simple cases of n, and n' will be discussed later.

C. PROPAGATION AT $\theta = \pi/2$.

In this case the only wave we consider is the extraordinary one since the ordinary will not propagate if

$$N_0^2 = P = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\Omega_p^2}{\omega^2} < 0\,. \tag{12.35}$$

Equation (12.44) is satisfied in the radio wave region and at the surface of a neutron star where the density is of the order of $N_e \simeq 10^{20}/\mathrm{cc} \simeq 10^{-4}$ g/cc The extraordinary mode has the following polarization vector

$$a(x) = (0, 1, 0)\,. \tag{12.36}$$

This form has been deduced from Equations (12.6) and (12.7), omitting the longitudinal component since our unit module's normalization is valid for pure transverse waves. In this case the quantity of interest is simply given by

$$E_\alpha j_\alpha^* + j_\alpha E_\alpha^* = \frac{8\pi^2}{\omega}\,|j_y(k, \omega)|^2\,\delta(\Lambda)\,.$$

Using Equation (12.39) we obtain

$$\begin{aligned}
\frac{\mathrm{d}I(\omega, \Omega)}{\mathrm{d}\omega\,\mathrm{d}\Omega} &= \frac{1}{2} I_0 \frac{\omega^2 N}{\sqrt{\varepsilon - \omega - n'\omega_H}} \int_0^\infty \frac{\mathrm{d}t}{(t + \tilde{\lambda})^2} \\
&\times \left\{ \frac{1}{(\omega - \omega_H)^2} [\sqrt{1 + n'}\,\mathscr{I}(n, n' + 1) + \sqrt{n}\,\mathscr{I}(n - 1, n)]^2 \right. \\
&+ \left. \frac{1}{(\omega + \omega_H)^2} [\mathscr{I}(n + 1, n')\sqrt{n + 1} + \sqrt{n'}\,\mathscr{I}(n' - 1, n)]^2 \right\}
\end{aligned} \tag{12.37}$$

$$\begin{aligned}
\tilde{\lambda} &= \frac{1}{2\omega_H} \{\sqrt{2(\varepsilon - n\omega_H)} - \gamma\sqrt{2(\varepsilon - \omega - n'\omega_H)}\}^2 \\
N^2 &= RL/S\,.
\end{aligned} \tag{12.38}$$

D. BREMSSTRAHLUNG EMISSION IN THE LOW QUANTUM NUMBER REGION

In this section we will give the explicit form of the radiation intensity for a few quantum numbers, namely, $n=0$, $n'=0$, $n'=1$, $n=1$, $n'=0$. Using Equations (12.31) and

(12.38), the following notation

$$\tilde{I}(n, n'; O, X) = \frac{1}{I_0} \frac{\mathrm{d}I(\omega, \Omega)}{\mathrm{d}\omega\, \mathrm{d}\Omega} \tag{12.39}$$

we obtain the emissivity along the direction of the field, $\theta = 0$:

$$\begin{aligned}
\tilde{I}(0, 0; O) &= \sqrt{L}\, \frac{\omega^2}{\sqrt{\varepsilon - \omega}} \frac{1}{E_4^2} C_1(\lambda) \\
\tilde{I}(0, 0; X) &= \sqrt{R}\, \frac{\omega^2}{\sqrt{\varepsilon - E}} \frac{1}{\omega_1^2} C_1(\lambda) \\
\lambda &= (2\omega_H)^{-1} [\sqrt{2\varepsilon} - \omega \sqrt{R, L} - \gamma \sqrt{2(\varepsilon - \omega)}]^2 \\
E_1 &= \omega - \omega_H - \tfrac{1}{2}\omega^2 R - \gamma\omega \sqrt{2R(\varepsilon - \omega)} \\
E_4 &= \omega + \omega_H + \tfrac{1}{2}\omega^2 L - \gamma\omega \sqrt{2L\varepsilon}
\end{aligned} \tag{12.40}$$

$$\begin{aligned}
I(1, 0; O) &= \sqrt{L}\, \frac{\omega^2}{\sqrt{\varepsilon - \omega}} \frac{1}{E_4^2} C_3(\lambda) \\
E_4 &= \omega + \omega_H + \tfrac{1}{2}\omega^2 L - \gamma\omega \sqrt{2L(\varepsilon - \omega_H)} \\
\lambda &\equiv (2\omega_H)^{-1} \{\sqrt{2(\varepsilon - \omega_H)} - \omega\sqrt{L} - \gamma\sqrt{2(\varepsilon - \omega)}\}^2 \\
I(1, 0; X) &= \sqrt{R}\, \frac{\omega^2}{\sqrt{\varepsilon - \omega}} \left\{ \frac{1}{E_1^2} [C_2 + C_3 - 2C_1] \right. \\
&\quad \left. + \frac{1}{E_2^2} C_2 + \frac{1}{E_1 E_2} [2C_1 - 2C_2] \right\} \\
E_1 &= \omega - \omega_H - \tfrac{1}{2}\omega^2 R - \gamma\omega \sqrt{2R(\varepsilon - \omega)} \\
E_2 &= \omega - \omega_H + \tfrac{1}{2}\omega^2 R - \gamma\omega \sqrt{2R(\varepsilon - \omega_H)} \\
\lambda &= (2\omega_H)^{-1} [\sqrt{2(\varepsilon - \omega_H)} - \omega\sqrt{R} - \gamma\sqrt{2(\varepsilon - \omega)}]^2
\end{aligned} \tag{12.41}$$

Analogously the emissivity perpendicular to the magnetic field, $\theta = \pi/2$ is given by:

$$\begin{aligned}
\tilde{I}(0, 0; X) &= \frac{1}{2} \sqrt{\frac{RL}{S}} \frac{\omega^2}{\sqrt{\varepsilon - \omega}} \left[\frac{1}{(\omega - \omega_H)^2} + \frac{1}{(\omega + \omega_H)^2} \right] C_1(\lambda) \\
\lambda &= (2\omega_H)^{-1} [\sqrt{2\varepsilon} - \gamma\sqrt{2(\varepsilon - \omega)}]^2 \\
\tilde{I}(1, 0; X) &= \frac{1}{2} \sqrt{\frac{RL}{S}} \frac{\omega^2}{\sqrt{\varepsilon - \omega}} \left[\frac{1}{(\omega - \omega_H)^2} + \frac{1}{(\omega + \omega_H)^2} \right] C_3(\lambda) \\
\lambda &= (2\omega_H)^{-1} [\sqrt{2(\varepsilon - \omega_H)} - \gamma\sqrt{2(\varepsilon - \omega)}]^2 \\
\tilde{I}(0, 1; X) &= \frac{1}{2} \sqrt{\frac{RL}{S}} \frac{\omega^2}{\sqrt{\varepsilon - \omega - \omega_H}} \left[\frac{1}{(\omega - \omega_H)^2} + \frac{1}{(\omega + \omega_H)^2} \right] C_3(\lambda) \\
\lambda &= (2\omega_H)^{-1} [\sqrt{2\varepsilon} - \gamma\sqrt{2(\varepsilon - \omega - \omega_H)}]^2
\end{aligned} \tag{12.42}$$

$$C_1(x) = e^x(1+x)E(x) - 1 \qquad C_2(x) = \frac{1}{x} - e^x E(x)$$
$$C_3(x) = 1 + x - (2x + x^2)e^x E(x) \qquad E(x) = \int_x^\infty \frac{e^{-s}}{s}\,\mathrm{d}s. \tag{12.43}$$

The present calculation differs from the previous one (CCFC69) in several significant ways. First, the Green's function used here is an exact one, while in the previous case the free particle Green's function has been used. Second, the plasma effect has been incorporated into our calculation. The final result, for example, Equation 12.40, can be interpreted easily. The factor $C_1(\lambda)$ arises from the Coulomb field of the nucleus, the factor ω^2 arises from the density of state of the photon, and finally, the factor E_4^{-2} comes from the Green's function and the factor $1/\sqrt{(\varepsilon-\omega)}$ comes from the density of the final state of the electron. As discussed earlier in an intense magnetic field an electron exhibits one dimensional behavior; instead of the usual expression $p^2\,\mathrm{d}p/\mathrm{d}E$, the density of the final state for a one dimensional particle is just $\mathrm{d}p/\mathrm{d}E \simeq 1/p$. The effect of the refractive medium on the photon is to alter the relation between ω and k, and this is taken into account throughout the calculation. This process has recently been applied to pulsar emission radiation (CHC70).

13. Astrophysical Applications

In the laboratory the generation of a steady state magnetic field up to a strength of 10^5 G is relatively easy. By using a generator and a capacitor (Kapitza magnet) an oscillatory field of strength of 10^6 G can be achieved. By using shaped charge and proper configurations a field of 10^7 G can be achieved. At a field of 10^7 G the Landau level spacing is about 1 eV, matter will disintegrate because of the enormous Zeeman level splitting the outer shell electrons suffer. As Regge pointed out, the chemistry of matter under intense fields is quite different from those with normal fields. In fact, effects of strong magnetic fields on biological systems have been detected.

Steady fields of the order of 10^7 G or greater can only be found in astrophysical bodies. According to the flux conservation law, the field of a current carrying plasma can increase as the square of the contraction ratio, C. In the case of white dwarfs, C is of the order of 100, and fields of the order of 10^7 can exist in white dwarfs. More drastically, in the case of neutron stars, C is of the order of 10^5 and fields up to 10^{14} G can exist. Such fields have been speculated in the past and have been ridiculed for being unrealistic. However, the discovery of pulsars (pulsed radio sourches) left no doubt that, not only neutron stars exist, but fields of the order of 10^{13} G or greater also exist.

However, much of the treatment of strong fields in the literature still follows the pattern of classical electrodynamics which we, among others including groups working actively in the Soviet Union, and Erber, have shown must fail under conditions currently associated with neutron stars. The most severe effect of strong magnetic fields on a plasma is the synchrotron radiation rate loss, which in a field of 10^{13} G and an electron energy of 1 MeV, is already as short as 10^{-19} sec. As the application of the physics of strong fields to neutron stars is still in progress, we will not go on into details here. (Ch69)

Appendix 1. Wave Function of an Electron in a Uniform, Constant Magnetic Field

A detailed derivation of the wave function is given in (JL49). We will report here only the result. The general feature is that the wave function contains or depends on a parameter which represents the degeneracy related with the location of the electron orbit in a magnetic field. The cylindrical coordinates ($x=r\cos\phi$, $y=r\sin\phi$, $z=z$) the wave function ψ has the form [S60].

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{A.1}$$

$$\psi_{1,3} = e^{-iEt/\hbar} \frac{e^{ik_z \cdot z}}{\sqrt{L}} \frac{e^{i(l-1)\phi}}{\sqrt{2\pi}} f_{1,3}(\varrho) \tag{A.2}$$

$$\psi_{2,4} = e^{-iEt/\hbar} \frac{e^{ik_z \cdot z}}{\sqrt{L}} \frac{e^{il\phi}}{\sqrt{2\pi}} f_{2,4}(\varrho) \tag{A.3}$$

$$f_{1,2,3,4} = \sqrt{2\gamma} \begin{pmatrix} C_1 & I_{n-1}^s(\varrho) \\ iC_2 & I_n^s \quad (\varrho) \\ C_3 & I_{n-1}^s(\varrho) \\ iC_4 & I_n^s \quad (\varrho) \end{pmatrix} \tag{A.4}$$

with

$$E = \eta mc^2 [1 + x^2 + 2nH/H_q]^{1/2} \quad \eta = \pm 1 \tag{A.5}$$

$$\begin{aligned} \gamma &= \tfrac{1}{2}\lambda_c^{-2} H/H_q \quad \varrho = \gamma r^2 \\ I_n^s(x) &= (n!\, s!)^{-1/2}\, e^{-x/2}\, x^{(n-s)/2}\, Q_s^{n-s}(x) \end{aligned} \tag{A.6}$$

$$Q_s^l(x) = (-)^s \sum_{j=0}^{s} (-)^j \frac{s!\, n!\, x^{s-j}}{j!\,(s-j)!\,(n-j)!}. \tag{A.7}$$

$n = 1 + s = 0, 1, 2 \ldots$ principal quantum number. $s = 0, 1, 2 \ldots$ the radial one and $1 = 0$, ± 1, $\pm 2 \ldots (-\infty < l < \infty)$ the azimuthal ones. The 4 coefficients C_k in (A.4) at this stage are completely arbitrary except for the normalization condition

$$\sum_{i=1}^{4} |C_i|^2 = 1. \tag{A.8}$$

Their determination requires the introduction of a new operator $\hat{O}$ which commutes with the initial Hamiltonian. Two choices are usually made (TBZ66).

(1) $\hat{O}=\boldsymbol{\sigma}\cdot\boldsymbol{\pi}$, $\boldsymbol{\pi}=\mathbf{p}-e/(c)\,\mathbf{A}$: this operator describes the projection of the spin on the direction of motion. The requirement that

$$\hat{O}\psi = \hbar\tilde{k}\tilde{\zeta}\psi \tag{A.9}$$

with $(\hbar c\tilde{k})^2=E^2-m^2c^4$ determine uniquely the four C_k. The eigenvalue $\tilde{\zeta}$ can take the value ± 1. The C_k comes out to be

$$\begin{aligned} &C_1 = \tilde{\zeta}\tilde{\alpha}\tilde{A} \qquad C_2 = \tilde{\alpha}\tilde{B} \qquad C_3 = \tilde{\beta}\tilde{A} \qquad C_4 = \tilde{\zeta}\tilde{\beta}\tilde{B} \\ &2\tilde{A}^2 = 1 + \tilde{\zeta}k_3/\tilde{k} \qquad 2\tilde{B}^2 = 1 - \tilde{\zeta}k_3/\tilde{k} \\ &2\tilde{\alpha}^2 = 1 + mc^2/E \qquad 2\tilde{\beta}^2 = 1 - mc^2/E \end{aligned} \tag{A.10}$$

(2) $\hat{O}=\Pi_{12}$

$$\begin{aligned} \Pi_{12} &= mc^2\,\sigma_3 + c\varrho_2\,(\boldsymbol{\sigma}\times\boldsymbol{\pi})_3 \\ \Pi_{12}\psi &= \hbar ck\zeta\psi\,. \end{aligned} \tag{A.11}$$

In this case $\zeta=\pm 1$ characterizes the state of the spin polarization relative to the direction of the magnetic field: $\zeta=1$ along the field and $\zeta=-1$ against the field. In this case we found

$$\begin{aligned} &C_1 = aA \qquad C_2 = -\zeta bB \qquad C_3 = bA \qquad C_4 = \zeta aB \\ &2A^2 = 1 + \zeta\lambda_c^{-1}/k \qquad 2B^2 = 1 - \zeta\lambda_c^{-1}/k \\ &2a = (1 + p_3c/E)^{1/2} + \zeta\,(1 - p_3c/E)^{1/2} \\ &2b = (1 + p_3c/E)^{1/2} - \zeta\,(1 - p_3c/E)^{1/2}\,. \end{aligned} \tag{A.12}$$

The wave function used in References (K54) and (FC69) correspond to the first choice. In (K54) the wave function is given Cartesian coordinates and in this form it was taken in (CC68a).

Appendix II. Green Function for an Electron in a Constant Magnetic Field

We shall quote here 3 forms for the Green function.

1. NONRELATIVISTIC CASE *(Zero temperature)* (KMH65)

$$\begin{aligned} G(r, r' \,|\, E) \equiv \lim_{\varepsilon\to 0} \sum_{n,\,p_z,\,s} \frac{\psi_{n,\,p_z,\,s}(r)\,\psi^{\dagger}_{np_zs}(r)}{E(np_z) - (E \pm i\varepsilon)} \equiv \frac{m}{2\pi_l{}^2\hbar^2} \\ \times \left\{ \pm \sum_{n=0}^{N-1} J_n(x, y-y', x')\,\lambda_n(E)\exp\left[\pm i\,|z-z'|\,\lambda_n^{-1}(E)\right] \right. \\ \left. - \sum_{n=N}^{\infty} J_n(x, y-y', x')\,\lambda_n(E)\exp\left[-\,|z-z'|\,\lambda_n^{-1}(E)\right] \right\} \end{aligned} \tag{A.13}$$

with

$$J_n = \exp\left\{\left(\frac{H/H_q}{2\lambda_c^2}\right) \times \left[i(x+x')(y-y') - \tfrac{1}{2}(x-x')^2 - \tfrac{1}{2}(y-y')^2\right]\right\} \mathscr{L}_n(\alpha) \tag{A.14}$$

$$\alpha \equiv \frac{H/H_q}{2\lambda_c^2}\left[(x-x')^2 + (y-y')^2\right]. \tag{A.15}$$

The quantity N is the smallest integer, not smaller than $E/(mc^2\theta) - 1$. When $|r-r' \ll \ll (H/H_q)^{1/2} \lambda_c^{-1}$. Equation (A.13) can be simplified to ($\hbar\omega_c \equiv mc^2 H/H_q$)

$$G(r-r' \mid E) = \frac{m}{2\pi\hbar^2}\left\{|r-r'|^{-1} + \frac{\hbar/l^2}{\{2m[(n+\frac{1}{2})\hbar\omega_c - E]\}^{1/2}} \pm i \sum_{n=0}^{N} \frac{\hbar/l^2}{\{2m[E-(n+\frac{1}{2})\hbar\omega_c]\}^{1/2}}\right\}$$

2. NONRELATIVISTIC CASE *(Finite temperature)* (L68)

The usual definition of the finite temperature Green function is

$$G(x, x') = \beta^{-1} \sum_n \exp[-i\omega_n(\tau - \tau')]\, G_n(x, x')$$

$$\omega_n = (2n+1)\pi\beta^{-1}, \qquad \beta^{-1} = \kappa T, \qquad n = 0, \pm 1, \pm 2, \ldots.$$

The function $G_n(x, x')$ can be shown to have the following general form

$$G_n(x, x') = \exp\left[-\frac{i}{2} m\omega_c (y+y')(x-x')\right] \bar{G}_n(\mathbf{x}-\mathbf{x}')$$

where the vector $\mathbf{x}-\mathbf{x}'$ has the following components

$$\mathbf{x}-\mathbf{x}' = [(\mathbf{x}-\mathbf{x}')_\perp, z-z'].$$

The function $\bar{G}_n(\mathbf{x}-\mathbf{x}')$ is now Fourier transformed

$$\bar{G}_n(\mathbf{x}-\mathbf{x}') = \int \frac{d^3p}{(2\pi)^3} \exp[i\mathbf{p}(\mathbf{x}-\mathbf{x}')]\, G_n(p)$$

and $G_n(p)$ is finally computed to be

$$G_n(p) = 2e^{-\eta/2} \sum_{N=0}^{\infty} \frac{(-)^N L_N(\eta)}{i\omega_n - E(Np_z)}$$

with

$$\eta \equiv \frac{2}{m\omega_c} (p_x^2 + p_y^2)$$

$$E(N, p_z) = \frac{1}{2m} p_z^2 + (N + \tfrac{1}{2}) \hbar\omega_c - \mu$$

L_N is the Laguerre polynomial of order N and μ is the chemical potential.

3. Relativistic case *(Zero temperature)* (KU64)

The first computation of the Green function in a constant magnetic field was performed by Schwinger in 1949 (Sc49). We will report here a more recent computation by Kaitna and Urban (KU64). The relativistic Green function satisfies the following differenticl equation

$$\left(i\partial_\nu \gamma^\nu + \frac{e}{\hbar} A_\nu \gamma^\nu - \lambda_c^{-1} \right) G(xy) = \delta^4 (x - y).$$

With the ansatz

$$G(xy) = \left(i\partial_\mu \gamma^\mu + \frac{e}{\hbar} A_\mu \gamma^\mu + \lambda_c^{-1} \right) \bar{G}(xy)$$

the Fourier transform of $\bar{G}(x, y)$ i.e.,

$$\bar{G}(p, q) = \frac{1}{(2\pi)^2} \int \exp(ipx) \exp(-iqy) \bar{G}(x, y) \, \mathrm{d}^4x \, \mathrm{d}^4y$$

is shown to be given by

$$G(p, q) = (32\omega_c^2\pi^4)^{-1} \delta(p_0 - q_0) \exp\left[\frac{i}{\omega_c} (p_1 q_2 - q_1 p_2) \right] Z^{-1/2 - a/2b} \times \begin{pmatrix} [\Gamma(1 + a/b)]^{-1/2} & w_{-(1/2)-(a/b), 0}(Z) & 0 \\ 0 & & \Gamma(a/b) \, w_{-a/b, 0}(Z) \end{pmatrix}$$

with

$$Z \equiv \omega_c^{-1} [(p_1 - q_1)^2 + (p_2 - q_2)^2], \quad a \equiv p_0^2 - p_3^2 - \lambda_c^{-2}, \quad b \equiv -4\omega_c.$$

List of Symbols

α	Sommerfeld's fine structure constant
$\alpha_x, \alpha_y, \alpha_z$	components of polarization vector
$\mathbf{a}$	polarization vector
A	mass number of a nucleus
$\tilde{A}$	vector potential operator
A_0	scalar Coulomb's potential
A	coefficient of plasma dispersion relation
$\mathbf{A}_1(\mathbf{r}, t)$	vector potential of the self consistent electromagnetic field in a plasma

A_μ	four vector potential
B	magnetic induction
B_q	critical magnetic field strength
C	coefficient of plasma dispersion relation
c	speed of light
c_v	specific heat at constant volume
D	plasma parameter as defined by Stix
$\delta_{\alpha\beta}$	Kronecker delta
$E \equiv E(p_z, n)$	total electron energy
$\mathbf{E}$	electric field vector
E_α, E_β	electric field components
$\varepsilon_{\alpha\beta}$	plasma dielectric tensor
ε_{11}, ε_{12}, ε_{23} etc.	are components of the plasma dielectric tensor
e^-	electron
e	charge of the electron
e_n^-	electron in the state n
ε	total electron energy (in a strong magnetic field) in units of rest energy of the electron mc^2

$$f_{\parallel} \equiv \frac{\partial}{\partial p_{\parallel}} f_0,\ f_{\perp} \equiv \frac{\partial}{\partial p_{\perp}} f_0$$

F	plasma parameter as introduced by Stix
$f(\mathbf{r}, \mathbf{p}, t)$	Wigner distribution function
$f_0(p)$	equilibrium distribution
$f_1(rp\lambda)$	perturbed first order Wigner function
$f(x)$	Fermi distribution, $x \equiv p/mc$
g_A	axial vector coupling constant
g_V	vector coupling constant
γ	electron total energy in units of mc^2
γ_μ, γ_5	Dirac matrix $\mu = 1, 2, 3, 4$
$\frac{\partial}{\partial x_\mu} \equiv \partial_\mu$	covariant derivature
$\mathbf{H}$	magnetic field vector
H_q	strength of the critical magnetic field
h	Planck's Constant
$\hbar$	rationalized Planck's Constant
$\tilde{I}$	radiation intensity
$\mathbf{j}$	electric current vector
$\tilde{j}_\alpha$	quantum mechanical current operator
K	Boltzmann's constant
K_H	conductive opacity coefficient
k	propagation vector
$\mathscr{L}$	Lagrangian

L	plasma Parameter introduced by Stix
l	attenuation coefficient
λ_H	thermal conductivity coefficient in magnetic field H
M	magnetization
$\lambda \equiv g_A/g_V$	magnetic moment of electron gas
M	
m	rest mass of the electron
M_i	rest mass of ion
μ_β	Bohr magnetron
μ	chemical potential
N	particle density
n_l	refractive index of magnetoactive plasma
n	quantum number
N_0	electron number density
n_+, n_-	refractive index for circularly polarized waves
N	index of refraction
$n_{\alpha\beta}$	tensor related to the dielectric tensor $\varepsilon_{\alpha\beta}$
$\eta_{xx}\ \eta_{xy}$ etc.	plasma parameters related to plasma dielectric tensor
ω	circular frequency
ω_n	level degeneracy factor in magnetic field
ω_C	cyclotron frequence for electron ($\omega_C \equiv \omega_H$)
ω_L	relativistic Larmor frequency
ω_p	plasma frequency for electron
Ω_p	plasma frequency for ion
Ω_H	cyclotron frequency for ion
Ω	volume
w_j	Fermi distribution function
w	electron energy in units of mc^2
$P_{xx}, P_{yy} \equiv P_\perp$	
$P_{zz} \equiv P_\parallel$,	
$P_{zz} \equiv P_\parallel$	Pressure along the magnetic field
$P_\perp$	Pressure perpendicular to magnetic field
$\mathbf{p}$	momentum vector of electron
p_x, p_y, p_z	components of electron momentum
$\omega_1, \omega_2, \omega_3, \omega_4$	spin dependent functions
Q	Heat flow
q	wave vector
$\varrho(\mathbf{r}, \mathbf{r}')$	equilibrium density matrix
$\varrho_6 \equiv 10^{-6}\varrho$	density in g/cm^3
ϱ_t	transverse electrical resistance
R_L	Larmor radius
S	s matrix
S	Stix's plasma parameter

s	spin quantum number
$T_{\alpha\beta}$	electromagnetic stress tensor
$T_{\mu\nu}$	energy momentum tensor
T	temperature (in K)
$T_0 \equiv \frac{mc^2}{K}$	relativistic temp
τ	half life of electron
U	energy density
v	electron velocity
$v_{\parallel}$	electron velocity parallel to magnetic field
$v_{\perp}$	electron velocity perpendicular to magnetic field
$\sigma_{\alpha\beta}$	electrical conductivity tensor
$\sigma_{\parallel}$	electrical conductivity along magnetic field
$\sigma_{\perp}$	electrical conductivity perpendicular to magnetic field
$\sigma_{\mu\nu}$	product of Dirac matrix $(= -i\gamma_\mu\gamma_\nu)$
z	atomic number
Z	grand partition function
χ	magnetic susceptibility
x	electron momentum in units of mc
ψ	wave function of electron
ϕ	scalar potential
θ	angle
$\theta \equiv H/H_q$	field strength parameter $(H_q \equiv m^2c^3/e\hbar = 4.414 \times 10^{13}$ G)
λ_B	de-Broglie wave length
$\lambda\!\!\!^{-}_C$	Compton wave length of the electron divided by 2π
ν	frequency
ν_m	minimum frequency of emission
ν	neutrino
$\bar{\nu}$	antineutrino

References

A58 Argyres, P. N.: 1958, *Phys. Rev.* **109**, 115.

A70 Adler, S. L., Bahcall, J. N., Callan, C. G. and Rosenbluth, M. N.: 1970, *Phys. Rev. Letters* **25**, 1061.

AA56 Argyres, P. N. and Adams, E. N.: 1956, *Phys. Rev.* **104**, 900.

AB65 Akhiezer, A. I. and Berestetsky, V. B.: 1965, *Quantum Electrodynamics*, Interscience Publ. Inc., New York, Ch. IV.

AH59 Adams, E. N. and Holstein, T. D.: 1959, *Phys. Chem. Solids* **10**, 254.

AM59 Auluck, F. C. and Mathur, V. S.: 1959, *Z. Astrophys.* **48**, 28.

AR69 Arunalsalam, V.: 1969, *Math. Phys.* **10**, 1305.

BC62 Brittin, W. E. and Chappel, W. R.: 1962, *Rev. Mod. Phys.* **34**, 620.

BPP69 Baym, G., Pethick, C., and Pines, D.: *Nature* **224**, 674.

C67 Cameron, R. C.: 1967, *The Magnetic and Related Stars*, Mono Book Corp.

Ca67 Cameron, A. G. W.: 1967, *Astrophys. J. Letters* **1**, 9.

C69 Canuto, V.: 1970, *Astrophys. J.* **159**, 641.

C71 Canuto, V.: 1971, *Astrophys. J.*, to be published.

CC68a Canuto, V. and Chiu, H. Y.: 1968, *Phys. Rev.* **173**, 1210.
CC68b Canuto, V. and Chiu, H. Y.: 1968, *Phys. Rev.* **173**, 1220.
CC68c Canuto, V. and Chiu, H. Y.: 1968, *Phys. Rev.* **173**, 1229.
CC68d Canuto, V. and Chiu, H. Y.: 1968, *Astrophys. J. Letters.* **153**, 157.
CC68e Chiu, H. Y. and Canuto, V.: 1968, *Phys. Rev. Letters* **21**, 2, 110.
CC69 Canuto, V. and Chiu, H. Y.: 1969, *Phys. Rev.* **188**, 2446.
CC70 Canuto, V. and Chiuderi, C.: 1970, *Phys. Rev.* **D1**, 8, 2219.
CCC70 Canuto, V., Chiuderi, C., and Chou, C. K.: 1970, *Astrophys. Space Sci.* **7**, 407.
CCCFC70 Canuto, V., Chiu, H. Y., Chou, C. K., and Fassio-Canuto, L.: 1970, *Phys. Rev.* **D2**, 2, 281.
CCCL69 Canuto, V., Chiu, H. Y., Chiuderi, C., and Lee, H. J.: *Nature* **225**, 47.
CCFC68 Chiu, H. Y., Canuto, V., and Fassio-Canuto, L.: 1968, *Phys. Rev.* **176**, 1438.
CCFC68a Canuto, V., Chiu, H. Y., and Fassio-Canuto, L.: 1969, *Astrophys. Space Sci.* **3**, 258.
CCFC69 Chiu, H. Y., Canuto, V., and Fassio-Canuto, L.: 1969, *Phys. Rev.* **185**, 1607.
CCH69 Canuto, V. and Chiuderi, C.: 1969, *Nuovo Cimento* **2**, 223.
CFC69 Chiu, H. Y. and Fassio-Canuto, L.: 1969. *Phys. Rev.* **185**, 1614.
CG70 Cameron, A.G.W. and Gentile, A.: private communication.
Ch66 Chiu, H. Y.: 1966, *Ann. Rev. Nucl. Sci.* **16**, 591.
Ch66a Chiu, H. Y.: 1966, in *Stellar Evolution* (ed. by R. F. Stein and A. G. W. Cameron), Plenum Press, New York.
Ch68 Chiu, H. Y.: 1968, *Stellar Physics*, Blaisdell Publ. Co., Waltham, Mass., Ch. 5.
CHC70 Chiu, H. Y. and Canuto, V.: 1971, *Astrophys. J.* **163**, 577.
CS70 Canuto, V. and Solinger, A.: 1970, *Astrophys. J. Letters* **6**, 141.
CV71 Canuto, V., Ventura, J.: 1971, *Nuovo Cimento*, to be published.
D28 Darwin, B. C.: 1928, *Proc. Roy. Soc.* **117**, 258.
E66 Erber, T.: 1966, *Rev. Mod. Phys.* **38**, 626.
FC69 Fassio-Canuto, L.: 1969, *Phys. Rev.* **187**, 2141.
G67 Giriat, W.: 1967, *Phys. Letters* **24A**, 515.
G69 Greenstein, G.: 1969, *Nature* **223**, 938.
GO69 Gunn, J. E. and Ostriker, J. P.: 1969, *Nature* **223**, 813.
Gr69 Greene, M. P., Lee, H. J., Quinn, J. J., and Rodriquez, S.: 1969, *Phys. Rev.* **177**, 1019.
H31 Huff, I. D.: 1931, *Phys. Rev.* **38**, 510.
HL69 Hubbard, W. B. and Lampe, M.: 1969, *Astrophys. J., Suppl.* **163**, 18.
HS61 Hamada, T. and Salpeter, E. E.: 1961, *Astrophys. J.* **134**, 683.
J62 Jackson, J. D.: 1962, *Classical Electrodynamics*, John Wiley & Sons, New York, p. 313.
J69 Jancovici, B.: 1969, *Phys. Rev.* **187**, 2275.
JL49 Johnson, M. H. and Lippmann, B. A.: 1949, *Phys. Rev.* **76**, 828.
K27 Kennard, E. H.: 1927, *Z. Physik* **44**, 263.
K54 Klepikov, N. P.: 1954, *JETP* **26**, 19.
K65 Kubo, R.: 1965, *Statistical Mechanics*, North-Holland Publ. Co., Amsterdam, pp. 278–280.
Ka64 Kaitna, R. and Urban, P.: 1964, *Nucl. Phys.* **56**, 518.
Ke64 Kelly, D. C.: 1964, *Phys. Rev.* **134**, A641.
Ke70 Kelly, D. C.: 1970, *Am. Phys. Soc. Bull.* **15** (11), No. 1, 124.
KF59 Kahn, A. H. and Frederiske, H. P. R.: 1959, *Solid State Phys.* **9**, 257.
KMH65 Kubo, R., Miyake, S. J., and Hashitsume, N.: 1965, *Solid State Phys.* **17**, 270.
L30 Landau, L.: 1930, *Z. Physik* **64**, 629.
L66 Landstreet, J. D.: 1967, *Phys. Rev.* **153**, 1372.
L68 Lee, H. J.: 1968, *Phys. Rev.* **166**, 459.
LCCC69 Lee, H. J., Canuto, V., Chiu, H. Y., and Chiuderi, C.: 1969, *Phys. Rev. Letters* **23**, 390.
LL60 Landau, L. D. and Lifshitz, E. M.: 1960, *Electrodynamics of Continuous Media*, Pergamon Press, Ch. VIII.
LL62 Landau, L. D. and Lifshitz, E. M.: 1962, *Classical Theory of Fields*, Pergamon Press, Ch. 3.
M41 Marshak, R. E.: 1941, *Ann. Acad. Sci.* **41**, 49.
M50 Mestel, L.: 1950, *Proc. Cambridge Phil. Soc.* **46**, 337.
MR67 Mestel, L. and Ruderman, M. A.: 1967, *Monthly Notices Roy. Astron. Soc.* **136**, 27.
P30 Page, L.: 1930, *Phys. Rev.* **36**, 444.

P130 Plesset, M. S.: 1930, *Phys. Rev.* **36**, 1728.
R28 Rabi, I. I.: 1928, *Z. Physik* **49**, 507.
R52 Robl, B.: 1952, *Acta Phys. Austriaca* **6**, 105.
R65 Ruderman, M. A.: 1965, *Rep. Progr. Phys.* **28**, 411.
Ra44 Rasetti, R.: 1944, *Phys. Rev.* **66**, 1.
S58 Skobov, V. G.: 1958, *JETP* **35**, 1315.
S60 Sokolov, A. A.: 1960, *Introduction to Quantum Electrodynamics*, U.S. Atomic Energy Commission: AEC-tr-4322, pp. 174–223.
S61 Salpeter, E. E.: 1961, *Astrophys. J.* **134**, 669.
S61 Stix, T. H.: 1962, *The Theory of Plasma Waves*, McGraw-Hill Book Co., New York.
S66 Shklovsky, I. S.: 1966, *Soviet Astron. AJ* **10**, 6.
Sa67 Sakurai, J. J.: 1967, *Advanced Quantum Mechanics*, Addison-Wesley Publ. Co., Reading, Mass., pp. 107–110.
Sc49 Schwinger, J.: 1949, *Phys. Rev.* **75**, 898.
Sh66 Shafranov, V. D.: 1966, in *Reviews of Plasma Physics* **3**, Consultants Bureau, New York.
ST68 Sokolov, A. A. and Ternov, I. M.: 1968, *Synchrotron Radiation*, Pergamon Press, New York.
TBBD69 Ternov, I. M., Bagrov, V. G., Bordovitsyn, V. A., and Dorofeev, O. F.: 1969, *Soviet Phys. Doklady*, No. 13, 1219.
TBR64 Ternov, I. M., Bagrov, V. G., and Rzaev, R. A.: 1964, *JETP* **19**, 225.
TBZ66 Ternov, I. M., Bagrov, V. G., and Zhukovskii, V.: 1966, *Moscow Univ. Phys. Bull.* **1**, 21.
UY30 Uhlembeck, G. E. and Young, L. A.: 1930, *Phys. Rev.* **36**, 1721.
W64 Woltjer, L.: 1964, *Astrophys. J.* **140**, 1309.
Wa47 Wannier, G. H.: 1947, *Phys. Rev.* **72**, 304.
Z64 Zyrianov, P. S.: 1964, *Fiz. Metal. Metalloved.* **18**, 161.
Z65 Ziman, J. M.: 1965, *Principles of the Theory of Solids*, Cambridge Univ. Press.